# LECTURE 12



Covectors and Vectors

I. Definition of a linear function/functional

II. Set of linear functions as a vector space

III. Dirac bracket notation

IV. Construction of linear functions

V. Coordinate functions as a basis for the dual space

Start reading and assimilating

"The Dual of a Vector Space: From the Concrete to the Abstract to the Concrete."

(In Four Lectures)

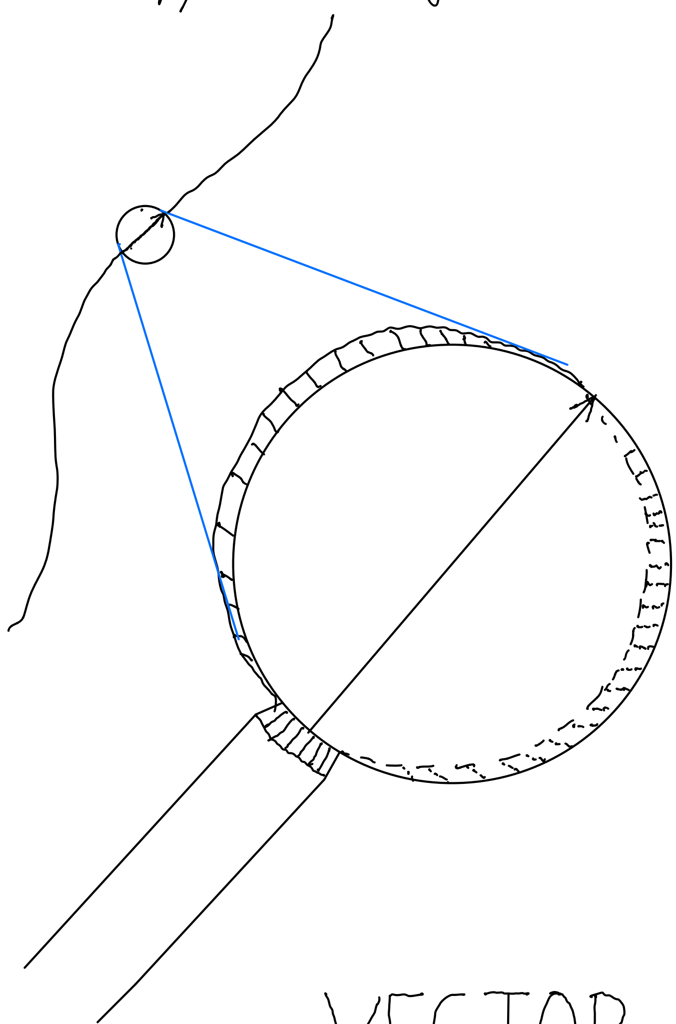
Also in MTW read Sections 2.5–2.7; 9.1–9.5

# Tensor Analysis: Algebra & Calculus

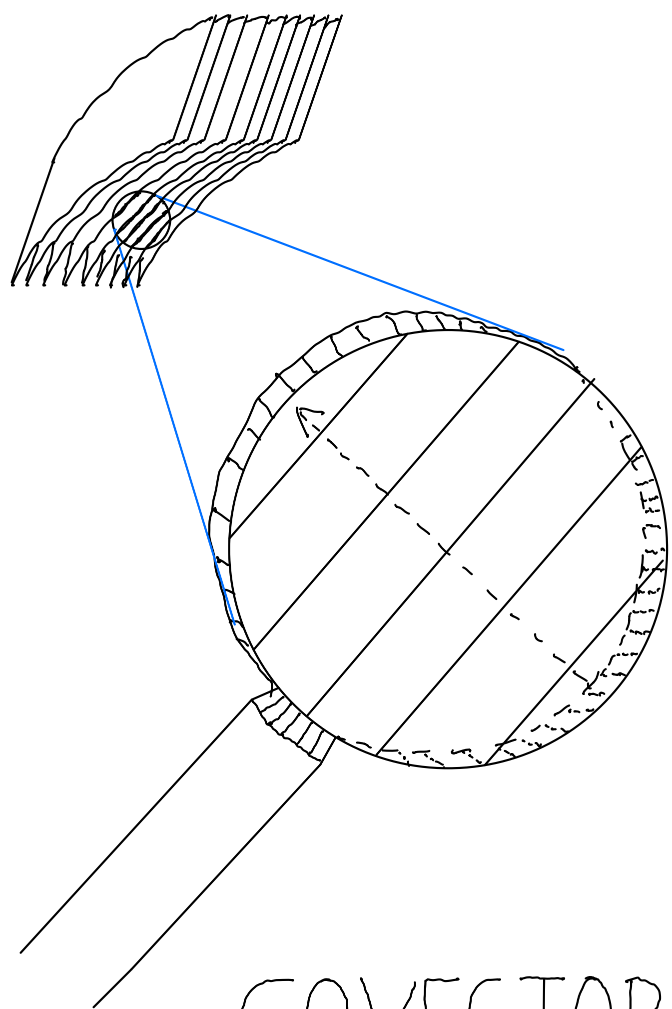


## Tensor Algebra

The basic building blocks for tensors are <u>vectors</u> and <u>covectors</u> (a.k.a. one-forms, duals, linear functions, linear fuctionals). The essential difference between them is that vectors arise in the context of curves, while covectors arise in the context of surfaces. A vector can be viewed as an infinitesimal curve segment, while a covector can be viewed as a local stack of parallel tangent level surfaces of a scalar-valued function.



VECTOR



COVECTOR

I. A covector is a derived concept. It depends on one's knowledge of what vectors are. More precisely, a covector is a concept which is summarized in terms of a linear function by means of the following

Definition ("Linear function")

Let $V$ be a vector space.

Consider a scalar-valued linear function $f$ defined on $V$:

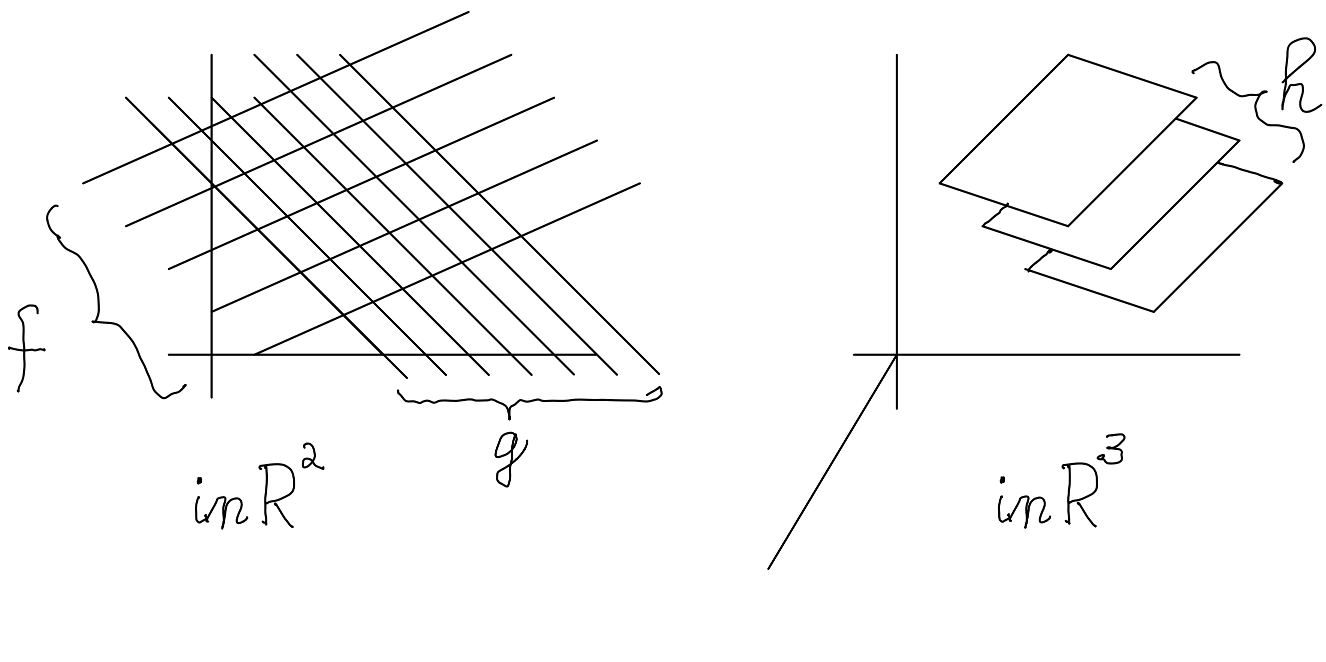
$$f: V \longrightarrow \text{reals}$$
$$x \rightsquigarrow f(x)$$

such that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \qquad \text{where } x, y \in V \text{ and } \alpha, \beta \in \mathbb{R}$$

Comment

Geometrically $f$ can be pictured as a <u>stack</u> of parallel planes, its <u>isograms</u> (= level surfaces) in $V$.




in $\mathbb{R}^2$ in $\mathbb{R}^3$

II. The set of linear functions form a vector space.

This fact is stated by the following

Theorem ("Vector space of duals")

Let $V^*$ be the set of all linear functions on $V$.

The set $V^*$ of such linear functions forms a vector space, the <u>dual space</u> of $V$.

Comment.

(a) The elements of this vector space $V^*$ of <u>duals</u> are called <u>covectors</u>

(b) That $V^*$ does indeed form a vector space is verified by observing that the collection of linear fuction satisfies the familiar ten properties.

If $f, g, h \in V^*$ and $\alpha, \beta \in \mathbb{R}$, then

1. $f+g$ defined by

   $(f+g)(x) = f(x) + g(x) \ \forall x \in V$ is also a linear function because it satisfies the definition on page 12.3. Thus one has closure under addition.

2. $f+g = g+f$
3. $(f+g)+h = f+(g+h)$
4. The zero element is the constant zero function.
5. The additive inverse of $f$ is $-f$.

and

1. $\alpha f$ defined by

   $(\alpha f)(x) = \alpha f(x) \ \forall V$ is also a linear function

2. $\alpha(\beta f) = (\alpha\beta) f$

3. $1 \cdot f = f$
4. $\alpha(f+g) = \alpha f + \alpha g$
5. $(\alpha+\beta) f = \alpha f + \beta f$

## III. Dirac's bracket notation

To capture the duality between $V$ and $V^*$ one introduces Dirac's <u>bra</u> c<u>ket</u> <u>notation</u>, which is familiar from quantum mechanics.

Thus, if $f$ is a linear function and $f(\vec{x})$ is its value at $\vec{x}$, then one also writes

$$f(\vec{x}) \equiv \langle f | \vec{x} \rangle \equiv \langle \underset{\sim}{f} | \vec{x} \rangle$$

One also says that the linear function (or functional) <u>operates</u> on the vector $\vec{x}$ and produces the scalar

$$\langle f | x \rangle .$$

To emphasize that $f$ is a linear "machine", one writes

$$f = \langle \underset{\sim}{f} | \ (= \langle f |)$$

for the covector and

$$\vec{x} = | \vec{x} \rangle \ (= | x \rangle)$$

for the vector. They combine to form

$$\langle \underset{\sim}{f} | \vec{x} \rangle = \langle f | x \rangle$$

## IV. Construction of a linear function.

There is a very direct way of constructing linear functions on $V$ once a basis, say $B = \{ e_i \}$ has been chosen for $V$.

The construction process proceeds as follows:

Every vector, such as $x$ and $y$, has a unique expansion in terms of this basis:

$$x = \alpha^1 e_1 + \cdots + \alpha^n e_n$$

$$y = \beta^1 e_1 + \cdots + \beta^n e_n$$

$$x + y = (\alpha^1+\beta^1) e_1 + \cdots + (\alpha^n+\beta^n) e_n$$

$$cx = c\alpha^1 e_1 + \cdots + c\alpha^n e_n$$

Note that $\alpha^1$ is uniquely determined by $x$

" that $\beta^1$ is uniquely " by $y$

" that $(\alpha^1+\beta^1)$ is uniquely " by $(x+y)$

" that $c\alpha^1$ is uniquely " by $cx$

Consequently, one has the following single valued relationships

$$\begin{aligned} x &\rightsquigarrow \alpha^1 \\ y &\rightsquigarrow \beta^1 \\ x+y &\rightsquigarrow \alpha^1+\beta^1 \\ cx &\rightsquigarrow c\alpha^1 \end{aligned}$$

Thus one has a <u>function</u>, call $\omega^1$, with the property that

$$\omega^1(x) = \alpha^1$$

$$\omega^1(y) = \beta^1$$

$$\omega^1(x+y) = \alpha^1+\beta^1$$

$$\omega^1(cx) = c\alpha^1$$

These properties imply that

$$\omega^1(x+y) = \omega^1(x) + \omega^1(y)$$

$$\omega^1(cx) = c\,\omega^1(x)$$

Consequently $\omega^1$ is a <u>linear</u> function on $V$.

The function $\omega^1$ is <u>the 1st coordinate function</u> determined by the basis $\{e_i\}_{i=1}^n$

Similarly one defines $\omega^j$ by

$$\omega^j(x) = \alpha^j \quad \text{for } j=2, \cdots, n.$$

Thus $\omega^j(x)$ is the $j^{th}$ coordinate of $x$ relative to the chosen basis. This statement holds for all $x$ in $V$. Thus one infers that, given the basis $\{e_i\}$,

$$\boxed{\omega^j \text{ is the } j^{th} \text{ coordinate function on } V.}$$

The evaluation of this function on each of the basis vectors

$$e_1 = 1\cdot e_1 + 0\cdot e_2 + \cdots + 0\cdot e_n$$

$$e_2 = 0\cdot e_1 + 1\cdot e_2 + \cdots + 0\, e_n$$

$$\vdots$$

$$e_n = 0\cdot e_1 + 0\cdot e_2 + \cdots + 1\, e_n$$

yields

$$\omega^1(e_1) = 1 \quad \omega^2(e_1) = 0 \quad \cdots \quad \omega^n(e_1) = 0$$

$$\omega^1(e_2) = 0 \quad \omega^2(e_2) = 1 \quad \cdots \quad \omega^n(e_2) = 0$$

$$\vdots$$

$$\omega^1(e_n) = 0 \quad \omega^2(e_n) = 0 \quad \cdots \quad \omega^n(e_n) = 1$$

or

$$\boxed{\omega^j(e_i) = \delta^j{}_i \equiv \begin{cases} 1 & \text{whenever } j=i \\ 0 & \text{whenever } j\neq i, \end{cases}}$$

Applying this algebraic relation to some vector $x = e_i x^i$ is, of course, the means for recovering its coordinate components:

$$\omega^j(x) = \omega^j(e_i x^i) = \omega^j(e_i)x^i = \delta^j{}_i x^i = x^j = j^{th} \text{ coordinate value of } x.$$

# V. Coordinate functions as a basis for $V^*$

Having been induced by the chosen basis $\{e_i\}$, the coordinate functions $\{\omega^j\}_{j=1}^n$ are not passive members of $V^*$. On the contrary, they act as – in fact, they constitute – a basis for $V^*$. This fact is expressed by the following

**Theorem** ("Basis for $V^*$")

Given: The basis $B = \{e_1, \ldots, e_n\}$ for $V$

Conclusion: The set of linear coordinate functions $\{\omega^j\}_{j=1}^n$, each of which satisfying

$$\omega^j(e_i) = \delta^j{}_i$$

is a basis for $V^*$.

Proof: Show that $\{\omega^j\}$ is a set which (i) is linearly independent and (ii) is a spanning set of $V^*$.

(i) Linear independence:

Consider any linear combination of the $\omega^j$'s, $\alpha_j \omega^j$, with property that it be the zero function on $V$. Evaluate it on each of the basis vector $e_k$. The result is that $\alpha_k = 0$.

(ii) Spanning property

Let $f \in V^*$. Evaluate $f$ at $x = e_i x^i$ and find that

$$
\begin{aligned}
f(x) &= f(e_i x^i) \\
&= f(e_i)\, x^i \\
&= f(e_i)\, \omega^i(x)
\end{aligned}
$$

This equality hold for all $x \in V$. Consequently,

$$\boxed{f = f(e_i)\, \omega^i}$$

That $V$ is the dual of $V^*$ whenever $V^*$ is the dual of $V$, and vice versa, i.e. $V$ and $V^*$ are duals of each other, follows from the following line of reasoning:

a) For any pair $(f, x) \in V^* \times V$ define the map

$$\langle \cdot | \cdot \rangle : V^* \times V \longrightarrow R$$

$$(f, x) \rightsquigarrow \langle f | x \rangle = f(x)$$

This map is bilinear because it is linear in each argument. Indeed

$$\langle \alpha f + \beta g | x \rangle = (\alpha f + \beta g)(x)$$
$$= \alpha \langle f | x \rangle + \beta \langle g | x \rangle$$

and

$$\langle f | ax + by \rangle = f(ax + by)$$
$$= a \langle f | x \rangle + b \langle f | y \rangle$$

b) Use Dirac's $\langle \cdot | \cdot \rangle$ in two ways:

1. For fixed $x$, let $\langle \cdot | x \rangle \equiv x(\cdot)$. This is a linear function on the vector space $V^*$

2. Conversely, any linear function on $V^*$ can be expressed this way. Indeed, let $\varphi$ be some linear function on the vector space $V^*$. Evaluate $\varphi$ on the basis elements $\omega^j$ and set $x = \varphi(\omega^j) e_j$. Then for any $f \in V^*$

$$\langle f | x \rangle = \varphi(\omega^j) \langle f | e_j \rangle = \varphi(\omega^j) f(e_j)$$
$$= \varphi(f(e_j)\, \omega^j)$$
$$= \varphi(f)$$

This means that $\langle \cdot | x \rangle$ is a linear function on $V^*$.

c) Combining 1. and 2. one has

$$x(f) = \langle f | x \rangle = f(x).$$

This holds for any $f$ and $x$ is $V^*$ and $V$ and thus expresses the "duality" between $V^*$ and $V$ in terms of Dirac's use of the symbol $\langle \cdot | \cdot \rangle$.