LECTURE 13

I. Addition and Meaning of Covectors

II. Metric on a Vectorspace

III. Natural Isomorphism Between Vectors and their Duals

Chew and assimilate the 2nd Lecture in "The Dual of a Vector Space...".
Then digest the 3rd and 4th Lecture.

I. The geometry of duals in an oblique coordinate system. Covector as a displacement density.

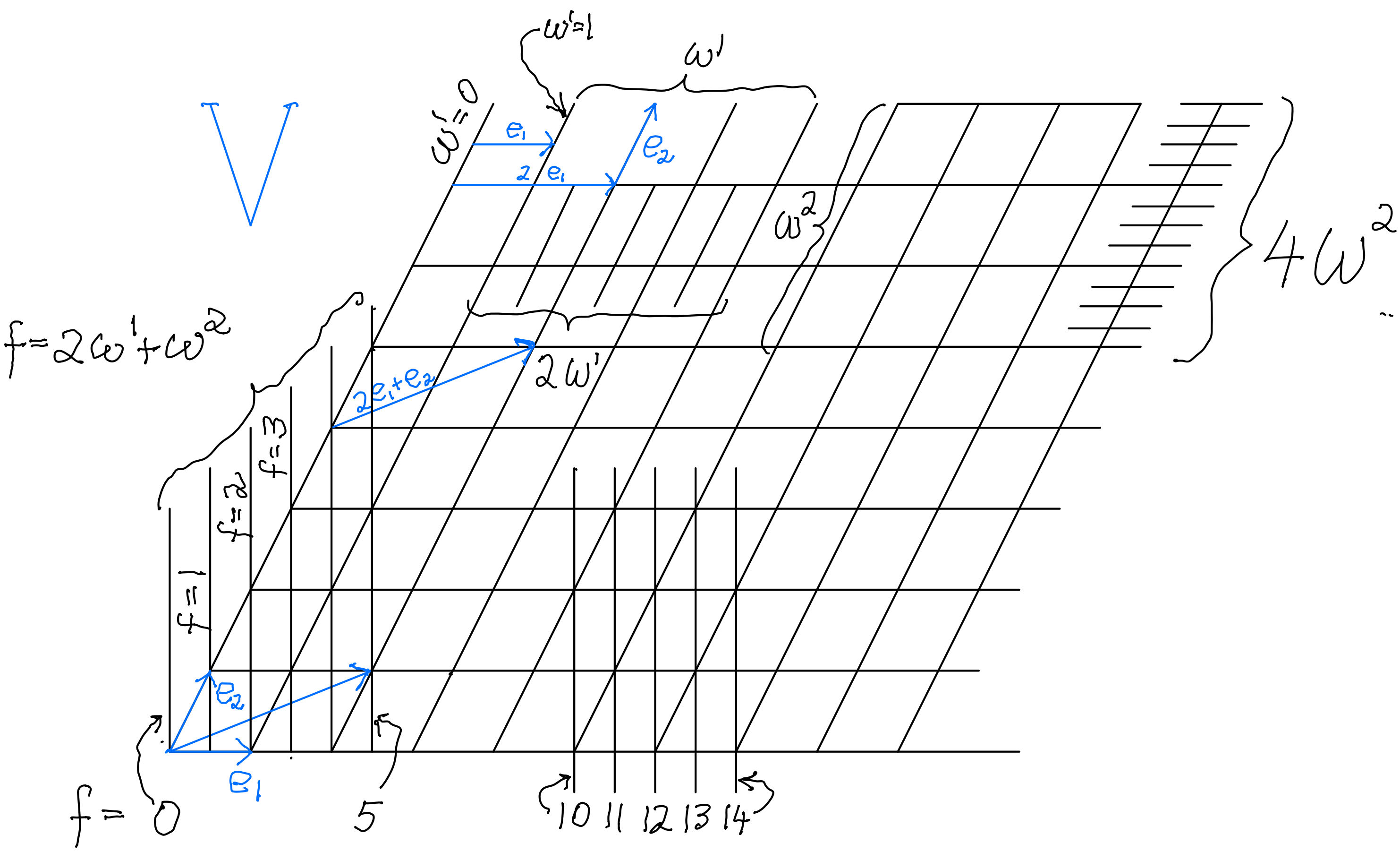

Figure 13.1: Addition of vectors and covectors. With $\{e_1, e_2\}$ as the basis for V, $\{\omega^1, \omega^2\}$ is it cobasis, i.e. the basis of V^*.

Given the covector $f=2\omega^1+\omega^2$, its geometrical properties in the given vector space V are captured by the fact that f ($\equiv \langle f| \equiv \underset{\sim}{f}$) refers to the density of its isograms in V. Thus

$$\left.\begin{aligned}\langle f|e_1\rangle &= 2 \;\Longrightarrow\; \langle f|\tfrac{e_1}{2}\rangle = 1\\ \langle f|e_2\rangle &= 1 \;\longrightarrow\; \langle f|e_2\rangle = 1\end{aligned}\right\} \Longrightarrow \text{The } f=1 \text{ isogram passes through } \tfrac{e_1}{2} \text{ and } e_2$$

implies that f's density into the direction of e_1 is 2, i.e. two units (of whatever physical quantity) per displacement standard e_1, and its density into the direction of e_2 is 1, i.e. one unit per displacement standard e_2.

The density of $f = \langle f|$ into the direction of the displacement $2e_1+e_2$ is

$$f(2e_1+e_2) \equiv \langle f | 2e_1+e_2 \rangle$$
$$= \langle 2\omega^1 + \omega^2 | 2e_1+e_2 \rangle$$
$$= 4+0+1+0 = 5 \quad (\text{"units"})$$

which, as shown in Figure 13.1, is the number of integral-valued isograms pierced by the vector $2e_1+e_2$.

II. Metric on a Vector space: Bilinear Functional

There no natural (i.e. unique an basis independent) isomorphism between V and V^*. However, if the vector space has inner product defined on it, then such an isomorphism is determined.

An inner product is implemented on a vector space by means of the following constellation of concepts.

Definition ("Bilinear Form")

Given: A vector space U and a vector space V.

A <u>bilinear functional</u> (or "form") on $U\times V$ (pairs of elements, one from U and one from V) is a function w;

$$w: \quad U\times V \longrightarrow \text{reals}$$
$$(x,y) \rightsquigarrow w(x,y)$$

with the properties

$$w(\alpha^1 x_1 + \alpha^2 x_2, y) = \alpha^1 w(x_1,y) + \alpha^2 w(x_2,y)$$
$$w(x, \beta^1 y_1 + \beta^2 y_2) = \beta^1 w(x,y_1) + \beta^2 w(x,y_2).$$

In other words, w is linear in each argument.

Definition ("Metric")

A metric (or inner product) is a bilinear g on $V\times V$ (pairs of elements in V)

$$g:\ (x,y) \rightsquigarrow g(x,y)$$

with the property $g(x,y) = g(y,x)$.

In other words, a real metric is <u>symmetric</u> in its two arguments

Comment.

1. If the metric were complex-valued, then the symmetry condition gets replaced by
$$g(x,y) = \overline{g(y,x)}$$
2. The metric $g(\ ,\ )$ is called a <u>scalar product</u> wherever g is positive, i.e. $g(x,x)>0\ \forall x\neq 0$.
3. In our development of tensor algebra we shall not insist that g be positive definite. We allow (for physical reasons) for the existence of non-zero vectors such that
$$g(x,x)=0 \quad \text{with} \quad x\neq 0.$$
The are "null vectors. We are forced to consider an inner product with such a null result if V is Minkowski spacetime.

Example ("Basis expansion of the metric")

Let
$$x = x^1e_1 + x^2e_2 + \cdots + x^n e_n$$
be a representation of a vector in terms of a basis $\{e_1,\cdots,e_n\}$ for V. Then

$$g(x^1e_1+x^2e_2+\cdots,\ y^1e_1+y^2e_2+\cdots) = x^1y^1g(e_1,e_1)+(x^2y^1+x^1y^2)\,g(e_1,e_2)+x^2y^2g(e_2,e_2)+\cdots$$

$$\text{notation}\left\{\begin{aligned} &\equiv x^1y^1\,e_1\cdot e_1 + (x^2y^1+x^1y^2)\ e_1\cdot e_2 + x^2y^2\ e_2\cdot e_2+\cdots \\ &\equiv x^1y^1\ g_{11} + (x^2y^1+x^1y^2)\ g_{12} + x^2y^2\ g_{22}+ \\ &\equiv x^i g_{ij} y^j \equiv [x]^t[g][y] \end{aligned}\right.$$

The coefficients $g_{ij} = e_i \cdot e_j = g(e_i, e_j)$ are the <u>components of the metric g with respect to the basis $\{e_1, e_2, \dots, e_n\}$.</u>

They are the inner products of all pairs of basis vectors.

III. Metric as a Natural Isomorphism Between V and V^*

A metric establishes a <u>natural</u>, i.e. basis independent isomorphism between the vector space V and its space of duals V^*.

To conserve notation, use the same symbol g to designate this correspondence.

A) It defining property is

$$g: V \longrightarrow V^*$$

$$x \rightsquigarrow \underset{\sim}{x} = g(x, \ ) \ (= "x\cdot \ ")$$

Here $\underset{\sim}{x}$ is that linear functional which, when operating on any vector $y \in V$, yields $g(x,y)$:

$$\underset{\sim}{x} \equiv "x\cdot" : V \longrightarrow R$$

$$y \rightsquigarrow \underbrace{\underset{\sim}{x}(y)}_{\langle x|y\rangle} = x \cdot y = g(x, y)$$

("Dirac notation")

B) Representation of g relative to the chosen basis $\{e_i\}$.

One can represent this g relative to any given basis as follows:

Proposition (Basis representation of g)

Let $\{e_i\}$ be a basis for V

Let $\{\omega^j\}$ be its dual basis for V^*

Then g is represented relative to the "tensor basis" $\{\omega^i \otimes \omega^j\}$ as the mapping

$$g = g(\ ,\ ) = g_{ij}\, \omega^i \otimes \omega^j : V \longrightarrow V^*$$

$$x \rightsquigarrow g(x,\ ) = g_{ij} \langle \omega^i | x \rangle \omega^j$$

$$e_k \rightsquigarrow g(e_k,\ ) = g_{kj}\, \omega^j$$

Comment

1. The tensor product $\otimes$ is the operation which is applied to a pair of linear maps, for example ω^i and ω^j, to obtain the bilinear map

$$\omega^i \otimes \omega^j : V \times V \longrightarrow R$$

$$(x, y) \rightsquigarrow \omega^i \otimes \omega^j (x, y) = \langle \omega^i | x \rangle \langle \omega^j | y \rangle$$
$$= x^i y^j$$

2. Use this bilinear map to obtain the components of g:

$$g(e_k, e_l) = g_{ij}\, \omega^i \otimes \omega^j (e_k, e_l)$$
$$= g_{ij} \langle \omega^i | e_k \rangle \langle \omega^j | e_l \rangle$$
$$= g_{ij}\, \delta^i_k\, \delta^j_l$$
$$= g_{kl}$$

More generally, evaluating g on the pair of vectors x and y, one obtain their inner product

$$g_{ij}\,\omega^i\otimes\omega^j(x,y) = g_{ij}\langle\omega^i|x\rangle\langle\omega^j|y\rangle$$
$$= g_{ij}\,x^i\,y^j$$
$$= e_i\cdot e_j\,x^i y^j$$
$$= (x^i e_i)\cdot(y^j e_j)$$
$$= x\cdot y$$

C) The metric g in its basis representation

$$\boxed{g = g_{ij}\,\omega^i\otimes\omega^j} \qquad (13.1)$$

maps the vector x in its basis representation

$$x = e_k x^k \in V$$

to its image $\underset{\sim}{x} = x_j\,\omega^j$ in V^*, where it is represented by

$$\underset{\sim}{x} = x^k g_{kj}\,\omega^j \qquad \in V^*$$

This mapping process takes direct advantage of the boxed Eq. (13.1) with the result

$$g_{ij}\,\omega^i\otimes\omega^j(e_k x^k,\cdot) = g_{ij}\underbrace{\langle\omega^i|e_k\rangle}_{\delta^i_k}x^k\,\omega^j(\cdot)$$
$$= x^k g_{kj}\,\omega^j(\cdot)$$
$$\equiv x^k g_{kj}\langle\omega^j| \qquad \in V^*$$

Thus, the coordinate components x_j of the image of $x = e_k x^k$ produced by g are obtained from x^k by <u>lowering</u> its indeces,

$$x_j = x^k g_{kj}.$$

This relation came, of course, from the mapping depicted in Figure 13.2 below:

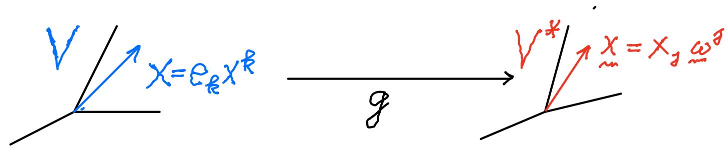

Figure 13.2: The metric g is a mapping between the given vector space V and its dual space V^*.