LECTURE 15

I. Tensor as a multilinear map.

II. Coordinate components of a tensor

III. Tensor product

In MTW read 3.2; Box 3.2; 3.5; 4.2-4.3.

I. Tensors are multinear maps, be they linear coming in the form of covectors, bilinear as they come in the form of metrics, or n-linear when they come in the form of a determinant.

Consider the $n \times n$ array of the components of n vectors in an n-dimensional vector space:

$$
\begin{array}{ll}
\vec{A}_1: & A_1^1 \ A_1^2 \ \cdots \ A_1^n \\
\vec{A}_2: & A_2^1 \ A_2^2 \ \cdots \ A_2^n \\
\vdots & \\
\vec{A}_n: & A_n^1 \ A_n^2 \ \cdots \ A_n^n \ .
\end{array}
$$

Its determinant,

$$
(\vec{A}_1, \vec{A}_2, \cdots, \vec{A}_n) \overset{\det}{\rightsquigarrow} \begin{vmatrix} A_1^1 & A_1^2 & \cdots & A_1^n \\ A_2^1 & A_2^2 & \cdots & A_2^n \\ \vdots & & & \\ A_n^1 & A_n^2 & \cdots & A_n^n \end{vmatrix} \equiv \det(\vec{A}_1, \vec{A}_2, \cdots, \vec{A}_n) \in R,
$$

is a map from n-tuples of n-dimensional vectors into the reals. It has the property of being linear in each of its arguments:

$$
\det(\vec{A}_1, \cdots, \alpha \vec{A}_i + \beta \vec{B}_i, \cdots, \vec{A}_n) = \alpha \det(\vec{A}_1, \cdots, \vec{A}_i, \cdots, \vec{A}_n) + \beta \det(\vec{A}_1, \cdots, \beta \vec{B}_i, \cdots, \vec{A}_n).
$$

The ubiquitous manifestations of this property are condensed into the following

Definition ("Multilinearity")

Let $V_1, V_2, \cdots, V_q$ be vector spaces. Then the map

$$\mathbb{H}: V_1 \times V_2 \times \cdots \times V_q \longrightarrow R$$

$$(v_1, v_2, \cdots, v_q) \rightsquigarrow \mathbb{H}(v_1, v_2, \cdots, v_q)$$

is said to be <u>multilinear</u> if it is linear in each of its arguments:

$$\mathbb{H}(v_1, v_2, \cdots, \alpha v_i + \beta w_i, \cdots, v_q) = \alpha \mathbb{H}(v_1, v_2, \cdots, v_i, \cdots, v_q) + \beta \mathbb{H}(v_1, v_2, \cdots, w_i, \cdots, v_q), \ \forall \ 1 \le i \le q$$

Nota bene: $(v_1, v_2, \cdots, v_q)$ is called a <u>q-tuple of vectors</u>.

This multilinearity property applies to vector spaces even when they have different dimensions. However, when they have the same dimension and refer to the same vector space, or its dual, then $\mathbb{H}$ is is called a tensor. This circumstance is condensed into the following

Definition ("Tensor")

Let

$$V_1 = \cdots = V_n = V^*$$

$$V_{n+1} = \cdots = V_{n+m} = V,$$

then the multilinear map

$$\mathbb{H}: \underbrace{V^* \times V^* \times \cdots \times V^*}_{\substack{n \text{ copies} \\ \text{of } V^*}} \times \underbrace{V \times V \times \cdots \times V}_{\substack{m \text{ copies} \\ \text{of } V}} \xrightarrow{\quad \mathbb{H} \quad} R$$

$$(\underbrace{\underset{\sim}{f}, \underset{\sim}{\sigma}, \cdots \underset{\sim}{\tau}}_{\substack{n \\ \text{covectors}}}, \underbrace{\mathbf{u}, \mathbf{v}, \cdots, \mathbf{w}}_{\substack{m \\ \text{vectors}}}) \overset{\mathbb{H}}{\rightsquigarrow} \mathbb{H}(\underset{\sim}{f}, \underset{\sim}{\sigma}, \cdots, \underset{\sim}{\tau}, \mathbf{u}, \mathbf{v}, \cdots, \mathbf{w})$$

is a <u>tensor of rank</u> $\binom{n}{m}$.

Here n and m are called the "contravariant rank" and the "covariant rank" of H. H maps $(n+m)$-tuples into the reals.

Comment

As a mnemonic for rembering the roles of n and m for a tensor of rank $\binom{n}{m}$, it may be helpful to view H as an animal that eats special food consisting of n covectors and m vector before it "spits out" a real number. This mnemonic will be mathematized in the next lecture when H will be represented in terms of "tensor products".

Examples of Tensors

Name	Symbol	Mapping	Rank
covector	$\underset{\sim}{\omega}$	$V \longrightarrow R$ $v \rightsquigarrow \underset{\sim}{\omega}(v) \equiv \langle \underset{\sim}{\omega} \mid v \rangle$	$\binom{0}{1}$
metric	g	$V \times V \longrightarrow R$ $(u, v) \rightsquigarrow g(u, v) = u \cdot v$	$\binom{0}{2}$
vector	v	$V^* \longrightarrow R$ $\underset{\sim}{\sigma} \rightsquigarrow v(\underset{\sim}{\sigma}) \equiv \langle \underset{\sim}{\sigma} \mid v \rangle$	$\binom{1}{0}$
inverse metric	g^{-1}	$V^* \times V^* \longrightarrow R$ $(\underset{\sim}{f}, \underset{\sim}{h}) \rightsquigarrow g^{-1}(\underset{\sim}{f}, \underset{\sim}{h})$	$\binom{2}{0}$

II. Coordinate Components of a Tensor

A vector has components relative to a vector basis.
A covector has components relative to the dual basis.
Quite generally,
a tensor has components relative to the tensor basis.

The basis-dependent components of a tensor are its projections onto tuples of covectors and vectors as specified by the following

Definition ("Tensor components relative to a given basis")

Let $\{e_i\}$ be a basis for V.

Let $\{\omega^j\}$ be its dual basis for V^*.

Then the numbers

$$H(\omega^{j_1}, \omega^{j_2}, \cdots, \omega^{j_n}, e_{i_1}, e_{i_2}, \cdots, e_{i_m}) \equiv H^{j_1 j_2 \cdots j_n}{}_{i_1 i_2 \cdots i_m}$$

are the <u>tensor components</u> of H relative to the given basis.

The number of such components is $(\dim V)^{n+m}$.

Examples of tensorial coordinate components.

1. Consider the covector $\underset{\sim}{\alpha}$. It is a tensor of rank $\binom{0}{1}$. Its coordinate components are

$$\underset{\sim}{\alpha}(e_k) = \alpha_k$$
$$= \text{components relative to } \{e_k\}.$$

2. Consider the vector u. It is a tensor of rank $\binom{1}{0}$. Its coordinate components are

$$u(\omega^k) = u^i e_i(\omega^k)$$
$$\equiv u^i \langle \omega^k | e_i \rangle = u^i \delta^k{}_i$$
$$= u^k$$

3. The metric g is a tensor of rank $\binom{0}{2}$. Its coordinate components are

$$g(e_k, e_\ell) = g_{k\ell}$$

Comments

1. The result of evaluating H on some arbitrary $n+m$ tuple $(\rho, \cdots, \tau, u, \cdots w)$ is an $n+m$ fold <u>sum of products</u> whose coefficients are H's tensor component:

$$H(\rho_{j_1} \underset{\sim}{\omega}^{j_1} \cdots, \tau_{j_n} \underset{\sim}{\omega}^{j_n}, u^{i_1} e_{i_1}, \cdots w^{i_m} e_{i_m}) = H^{j_1 \cdots j_n}{}_{i_1 \cdots i_m} \rho_{j_1} \cdots \tau_{j_n} u^{i_1} \cdots w^{i_m}.$$

Note the error correction code which is built into Einstein summation convention:

a) the summation ("dummy") indeces appear only in pairs.

b) the distinction of upper vs lower indices is to be observed with rigid rigour.

III. Tensor Product

In 3-d Euclidean space consider a vector $\vec{v}$ rotating with angular velocity $\vec{\omega}$ around a given axis.

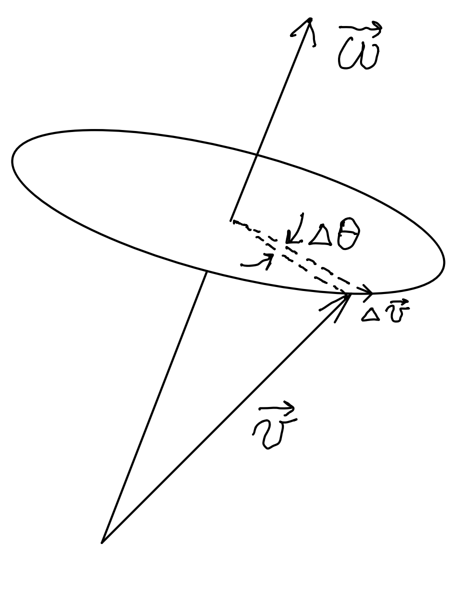

Figure 15.1: Vectorial change $\Delta\vec{v}$ in $\vec{v}$ due to rotation around $\vec{\omega}$. The tip of vector $\vec{v}$ moves in the plane perpendicular to $\vec{\omega}$, and the tip's angle of rotation during the time interval Δt is $\Delta\theta = |\vec{\omega}|\Delta t$.

The vectorial change $\Delta\vec{v}$ of $\vec{v}$ during the time interval Δt is

$$\Delta\vec{v} = \Delta t\ \vec{\omega}\times\vec{v}$$

In term of orthonormal basis vectors this cross product is

$$\Delta\vec{v} = \Delta t \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \omega^1 & \omega^2 & \omega^3 \\ v^1 & v^2 & v^3 \end{vmatrix} .$$

Expand this determinant in terms of the orthonormal basis vectors and find

$$\Delta\vec{v} = \Delta t\left[-\omega^1(\vec{e}_2 v^3 - \vec{e}_3 v^2) + \omega^2(\vec{e}_1 v^3 - e_3 v^1) - \omega^3(\vec{e}_1 v^2 - \vec{e}_2 v^1)\right].$$

Express the components of $\vec{v}$ in terms of inner products:
$e_k \cdot v = e_k \cdot e_i v^i = \delta_{ki} v^i = v^k$. Applying this yields

$$\Delta v = \Delta t\left[-\omega^1(\vec{e}_2\otimes\vec{e}_3 - \vec{e}_3\otimes\vec{e}_2) + \omega^2(\vec{e}_1\otimes\vec{e}_3 - \vec{e}_3\otimes\vec{e}_1) - \omega^3(\vec{e}_1\otimes\vec{e}_2 - \vec{e}_2\otimes\vec{e}_1)\right]\cdot\vec{v}. \tag{15.1}$$

The bivector $[\cdots]$ is a linear combination of "tensor products." There are three of them. The difference $(\vec{e}_2\otimes\vec{e}_3 - \vec{e}_3\otimes\vec{e}_2)$ generates rotation is the plane spanned by $\vec{e}_2$ and $\vec{e}_3$. The coeffcient $-\Delta t\,\omega^1$ is the amount of this rotation, and similarly for the other pairs of spanning vectors. The sum total in $[\cdots]$ of Eq. (15.1) is the rotation per unit time in the plane perpendicular to the rotation axis.

The mathematical generalization of those "tensor products is given by the following

Definition ("Tensor Product")

Let $\vec{a}, \vec{b}, \ldots, \vec{c} \in V$

and $\underset{\sim}{\alpha}, \underset{\sim}{\beta}, \ldots, \underset{\sim}{\gamma} \in V^*$

The multilinear map

$$\underbrace{\vec{a} \otimes \vec{b} \otimes \cdots \otimes \vec{c}}_{n} \otimes \underbrace{\underset{\sim}{\alpha} \otimes \underset{\sim}{\beta} \otimes \cdots \otimes \underset{\sim}{\gamma}}_{m} :$$

$$V^* \times V^* \times \cdots \times V^* \times V \times V \times \cdots \times V \longrightarrow R$$

$$(\underset{\sim}{\sigma}, \underset{\sim}{\rho}, \ldots, \underset{\sim}{\tau}, \vec{u}, \vec{v}, \ldots, \vec{w}) \rightsquigarrow \langle \underset{\sim}{\sigma} | \vec{a} \rangle \langle \underset{\sim}{\rho} | \vec{b} \rangle \cdots \langle \underset{\sim}{\tau} | \vec{c} \rangle \langle \underset{\sim}{\alpha} | \vec{u} \rangle \langle \underset{\sim}{\beta} | \vec{v} \rangle \cdots \langle \underset{\sim}{\gamma} | \vec{w} \rangle$$

is the <u>tensor product</u> $\vec{a} \otimes \cdots \otimes \underset{\sim}{\gamma}$. It is a tensor of rank $\binom{n}{m}$.