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I. Tensorial Basis Expansion

(16.2)

The basic mathematical building blocks of tensors are vectors and covectors. Tensor products of these elements results in new multilinear maps, i.e. tensors. Having generated $\binom{n}{m}$ rank tensors one can take linear combinations as dictated physical and geometrical considerations. Of these the most fundamental is to represent a tensor relative to a given basis. This task is achieved by the following

Proposition ("Basis representation of a tensor")

Given:

(i) a basis $\{e_i\}$ for V and its corresponding dual basis $\{\omega^j\}$ for V^*.

(ii) a tensor $\mathbb{H}$ of rank $\binom{n}{m}$.

Conclusion:

$$\boxed{\mathbb{H} = H^{j_1 \cdots j_n}{}_{i_1 \cdots i_m}\, e_{j_1} \otimes \cdots \otimes e_{j_n} \otimes \omega^{i_1} \otimes \cdots \otimes \omega^{i_m}} \qquad (16.1)$$

In this representation $\{ e_{j_1} \otimes \cdots \otimes e_{j_n} \otimes \omega^{i_1} \otimes \cdots \otimes \omega^{i_m} \}$ is the set of basis elements for $\binom{n}{m}$ tensors. The set $\{ H^{j_1 \cdots j_n}{}_{i_1 \cdots i_m} \}$ refers to the coordinates of $\mathbb{H}$ relative to this basis.

The validation of this representation consist of that the value of the l.h.s. equals that of the r.h.s. for all $(n+m)$-tuples of covectors and vectors.

To concretize this line of reasoning, apply it to an archetypical tensor, one of rank $\binom{1}{1}$:

$$\mathbb{H} : V^* \times V \rightarrow R$$

for which one must show that

$$\boxed{\mathbb{H} = H^i{}_j\, e_i \otimes \omega^j} \qquad (16.2)$$

The validation consists of showing that

$$\mathbb{H}(\underset{\sim}{\sigma}, \vec{v}) = H^i{}_j\, e_i \otimes \omega^j(\underset{\sim}{\sigma}, \vec{v}) \qquad \forall\ (\underset{\sim}{\sigma}, \vec{v}) \in V^* \times V, \qquad (16.3)$$

i.e., for all $\underset{\sim}{\sigma} \in V^*$ and $\vec{v} \in V$.

$\mathbb{H}$ is linear. Thus, it suffices to validate equality for all $(\omega^k, e_\ell) \in V^* \times V$. This is a two step process:

(i) Observe that as in the Definition on page 15.5,

$$\mathbb{H}(\omega^k, e_\ell) = H^k{}_\ell \qquad (16.4)$$

is the $(k,\ell)^{th}$ component of $\mathbb{H}$ relative to the given basis.

(ii) On the r.h.s. one has

$$H^i{}_j \, e_i \otimes \omega^j (\omega^k, e_\ell) = H^i{}_j \, e_i(\omega^k)\, \omega^j(e_\ell)$$

$$= H^i{}_j \underbrace{\langle \omega^k | e_i \rangle}_{\delta^k{}_i} \underbrace{\langle \omega^j | e_\ell \rangle}_{\delta^j{}_\ell}$$

$$= H^k{}_\ell \qquad (16.5)$$

Equations (16.4) and (16.5) hold for all pairs of basis elements (ω^k, e_ℓ), and, because of linearity of $\mathbb{H}$, Eq. (16.3) holds for all pairs $(\underset{\sim}{\sigma}, \vec{v})$. Thus Eq. (16.2) is a valid tensorial basis expansion indeed.

II. Examples

1. Metric tensor:

$$g = g_{ij}\, \omega^i \otimes \omega^j$$

2. Inverse metric tensor:

$$g^{-1} = g^{ij}\, e_i \otimes e_j$$

3. Cartan's unit tensor:

$$dP = \delta^i{}_j \, e_i \otimes \omega^j$$

$$= e_i \otimes \omega^i$$

4. <u>Totally</u> <u>antisymmetric</u> (Levi-Civita) <u>tensor</u> (a.k.a. <u>Volume</u> <u>tensor</u> in n dimensions):

$$\boldsymbol{\epsilon} = \epsilon_{i_1 \cdots i_n}\, \omega^{i_1} \otimes \cdots \otimes \omega^{i_n}$$

where $\epsilon_{i_1\cdots i_n}$ is the totally antisymmetric (Levi-Civita) symbol,

$$\epsilon_{i_1\cdots i_n} = \begin{cases} 0 & \text{if any pair of indices are the } \underline{\text{same}} \\ +\epsilon_{1\cdots n} & \text{if } i_1\cdots i_n \text{ is an } \underline{\text{even}} \text{ permuation of } 1,\dots,n \\ -\epsilon_{1\cdots n} & \text{if } i_1\cdots i_n \text{ is an } \underline{\text{odd}} \text{ permuation of } 1,\dots,n \end{cases}$$

An equivalent, but basis ("frame") indepenzdent definition is

$$\epsilon(\vec{A}_1,\cdots,\vec{A}_n) = \det \begin{vmatrix} \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \cdots & \omega^n(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \cdots & \omega^n(\vec{A}_2) \\ \vdots & & & \\ \omega^1(\vec{A}_n) & \omega^2(\vec{A}_n) & \cdots & \omega^n(\vec{A}_n) \end{vmatrix} = \begin{array}{l}\text{volume of a} \\ \text{parallelopiped} \\ \text{subtended by} \\ \{\vec{A}_i\}_{i=1}^n \text{ in } R^n\end{array}$$

III. Tensor Space

Tensors of rank $\binom{n}{m}$ can be added; they can be multiplied by scalars. This feature was already implicicit in Eq.(16.1), the basis reprentation of a tensor and the examples on pages 16.3 and 16.4. More formally on has the following

Proposition ("Tensor Space")

Tensors of rank $\binom{n}{m}$ form a vector space. This space is denotes by the $\underline{\text{tensor space}}$ $\underbrace{V\otimes\cdots\otimes V}_{n\text{ factors}}\otimes\underbrace{V^*\otimes\cdots\otimes V^*}_{m\text{ factors}}$.

A typical element H, an $\binom{n}{m}$ rank tensor, is evaluated on any of the elements of $V^*\times\cdots\times V^*\times V\times\cdots\times V$, the set of $(n+m)$-tuples.

IV. New Tensors

1. "Raising" and "lowering" tensor indices.

Consider a metric g.

$$V \xrightarrow{g} V^*$$

$$u = u^i e_i \overset{g}{\rightsquigarrow} g(u) = \underset{\sim}{u} = u_j \omega^j \in V^*$$

In transforming vectors into covectors, the effect of this metric on their expansion coefficients,

$$\{u^i\}_{i=1}^n \overset{g}{\rightsquigarrow} \{u_j = g_{ji} u^i\}_{j=1}^n ,$$

is to lower their indices for the creation of the covector $\underset{\sim}{u}$.

This process is generalized to tensors by means of the following

Proposition ("Lowering of indices)

The metric g lowers the indices of tensors

$$g : \underbrace{V \otimes \cdots \otimes V}_{n \text{ factors}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{m \text{ factors}} \longrightarrow \underbrace{V \otimes \cdots \otimes V}_{(n-1) \text{ factors}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{(m+1) \text{ factors}}$$

$$\binom{n}{m} \text{ tensors} \longrightarrow \binom{n-1}{m+1} \text{ tensors}$$

Applied to an $\binom{n}{m}$ tensor this process consists of the following transformation

$$H = \quad e_{j_1} \cdots e_{j_{n-1}} \boxed{e_{j_n}}\, \omega^{i_1} \cdots \omega^{i_m} \rightsquigarrow$$

$$H^{j_1 \cdots j_{n-1}}{}_{j_n i_1 \cdots i_m}\, e_{j_1} \cdots e_{j_{n-1}} \boxed{\omega^{j_n}}\, \omega^{i_1} \cdots \omega^{i_m}$$

The correspondingly coordinate components have their n^{th}

superscript (j_n) lowered:

$$\{H^{j_1\cdots j_{n-1}j_n}{}_{i_1\cdots i_m}\} \overset{g}{\rightsquigarrow} \{H^{j_1\cdots j_{n-1}k}{}_{i_1\cdots i_m}\, g_{kj_n}\} \equiv \{H^{j_1\cdots j_{n-1}}{}_{j_n i_1\cdots i_m}\}$$

2. Contraction of a tensor

Regardless of the nature of the metric on V, one can lower the rank of a tensor by means of the contraction map C its properties are condensed into the following

Definition ("Contraction of a tensor")

The contraction map C is a transformation on a tensor in which the one of the superscripts of it components get equated to one of their subscripts before one sums over the pair of equated indices:

$$C: \quad \binom{n}{m} \text{ tensors} \longrightarrow \binom{n-1}{m-1} \text{ tensors}$$

$$\{H^{j_1\cdots j_{n-1}j_n}{}_{i_1\cdots i_m}\} \rightsquigarrow \{H^{j_1\cdots j_{n-1}k}{}_{kj_2\cdots j_n}\}$$