LECTURE 17

I. Flux Vector

II. Flux Tensor

III. Generalization to 4-d spacetime

The process of mathematizing experiments and observations of the physical world are concretized among others by formulating the flow of matter in term of the Levi-Civita tensor. The result of this formulation is the flux tensor:

I. The Flux Vector

Consider an element of moving fluid consisting particles

(i) having uniform velocity

$$\vec{v} = v^i e_i = \left[\frac{(\text{displacement})}{(\text{time})}\right],$$

uniform particle density

$$N = \left[\frac{(\text{number})}{(\text{volume})}\right],$$

and passing through some chosen area spanned by $\vec{A}_1$ and $\vec{A}_2$ and hence

(ii) occupying during time Δt the interior of the parallelogram spanned by

$$\vec{A}_1, \vec{A}_2, \text{ and } \vec{v}\Delta t.$$

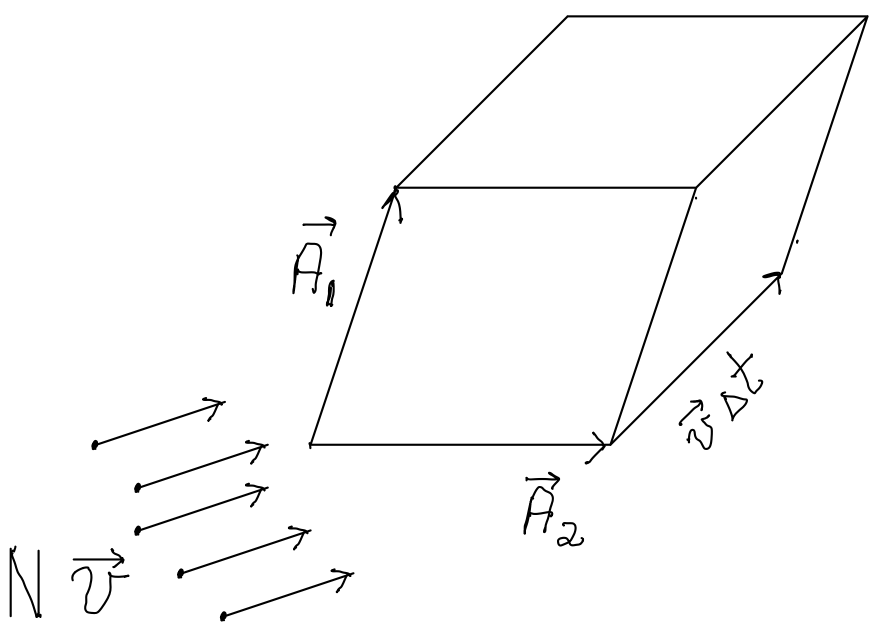

Figure 17.1: Parallelogram spanned by $\vec{A}_1, \vec{A}_2, \vec{v}\Delta t$ during time Δt.

The particle current, i.e., the measured number of particles (crossing the area $\vec{A}_1\times\vec{A}_2$) per unit time Δt, is $N\vec{v}\cdot\vec{A}_1\times\vec{A}_2$. In particular

$$\underbrace{\frac{1}{\Delta t}\cdot\overbrace{\underbrace{N}_{\text{"particle flux"}}(\vec{v}\Delta t)\cdot\underbrace{\vec{A}_1\times\vec{A}_2}_{\text{"area"}}}^{\substack{\text{"\# of particles}\\ \text{in volume}\\ (\vec{v}\Delta t)\cdot\vec{A}_1\times\vec{A}_2\text{"}}}} = \underbrace{\left[\frac{(\text{particles})}{(\text{area})(\text{time})}\right]}_{\text{"particle flux"}}(\text{area}) \qquad (17.1)$$

This construction on the l.h.s. of Eq. (17.1) is the particle current

$$\boxed{\frac{d(\#)}{dt} = N\vec{v}\cdot\vec{A}_1\times\vec{A}_2}\,, \qquad (17.2)$$

the number of particles per unit time crossing the area spanned by $\vec{A}_1$ and $\vec{A}_2$. The product of the particle density N and the common velocity $\vec{v}$ is the particle flux vector,

$$\boxed{N\vec{v}\equiv \boldsymbol{j} = \text{"particle flux vector"}} \qquad (17.3)$$

II. The Flux Tensor

The above particle current system consists of

(i) the element of fluid with its velocity $\vec{v}$ and density N, which are the given properties of the fluid, and of

(ii) the area $\vec{A}_1\times\vec{A}_2$, which is chosen.

Consequently, one must view the above expression (17.1) as a linear mapping of $V\times V$ into R. This observation leads to the definition of the particle flux $*\boldsymbol{j}$:

$$*\boldsymbol{j}:\ (\vec{A}_1,\vec{A}_2)\rightsquigarrow *\boldsymbol{j}(\vec{A}_1,\vec{A}_2) = N\vec{v}\cdot\vec{A}_1\times\vec{A}_2 \quad \left(=\frac{d(\#)}{dt}\right) \qquad (17.4)$$

Comment

17.4

The definition of $*j$ in Eq.(17.4) is deficient. This is because the mathematization of the number of particles in an element of fluid in terms of a vector crossproduct is restricted to 3-dimensional space. It does not generalize to 4-dimensional spacetime. However, its representation in terms of the corresponding determinant,

$$N\vec{v}\cdot\vec{A}_1\times\vec{A}_2 = N\det\begin{vmatrix} v^1 & v^2 & v^3 \\ A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \end{vmatrix} \qquad (17.5)$$

$$= N\det\begin{vmatrix} v^1 & v^2 & v^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix} \qquad (17.6)$$

does not suffer from this deficiency. Indeed, the following line of reasoning in three dimensions is readily extended to four dimensions.

(i) Apply the expansion of the determinant in Eqs(17.5)-(17.6) in terms of tensor products to Eq.(17.4) on page 17.3 and obtain

$$\begin{aligned} *j(\vec{A}_1,\vec{A}_2) = \{ & N v^1(\omega^2\otimes\omega^3-\omega^3\otimes\omega^2) \\ & + N v^2(\omega^3\otimes\omega^1-\omega^1\otimes\omega^3) \\ & + N v^3(\omega^1\otimes\omega^2-\omega^2\otimes\omega^1)\}(\vec{A}_1,\vec{A}_2) \end{aligned}$$

(ii) Introduce the antisymmetric wedge product $\omega^2\wedge\omega^3 \equiv \omega^2\otimes\omega^3-\omega^3\otimes\omega^2$; more generally,

$$\omega^j\otimes\omega^k-\omega^k\otimes\omega^j \equiv \omega^j\wedge\omega^k = -\omega^k\wedge\omega^j$$

and the totally antisymmetric Levi-Civita symbol in three dimensions

$$\epsilon_{ijk} = \begin{cases} 0 & \text{whenever any two indeces coincide} \\ \epsilon_{123} & \text{whenever } (ijk) \text{ is an \underline{even} permutation of } (123) \\ -\epsilon_{123} & \text{whenever } (ijk) \text{ is an \underline{odd} permutation of } (123). \end{cases}$$

This leads to

$$*\mathfrak{j}(\vec{A}_1,\vec{A}_2)=\Big\{\frac{N}{2}\Big[\; v^1\epsilon_{123}\,\omega^2\wedge\omega^3+v^2\epsilon_{231}\,\omega^3\wedge\omega^1+v^3\epsilon_{312}\,\omega^1\wedge\omega^2+$$

$$+\,v^1\epsilon_{132}\,\omega^3\wedge\omega^2+v^2\epsilon_{213}\,\omega^1\wedge\omega^3+v^3\epsilon_{321}\,\omega^2\wedge\omega^1\Big]\Big\}\,(\vec{A}_1,\vec{A}_2)$$

$$=N\,v^i\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!\;\;(\vec{A}_1,\vec{A}_2), \qquad (17.7)$$

which holds for all $(\vec{A}_1,\vec{A}_2)\in V\times V$.

(iii) Suppressing explicit reference to its arguments, the particle flux is

$$\boxed{*\mathfrak{j}=N\,v^i\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!} \qquad \text{"particle flux 2-form"} \qquad (17.8)$$

Comments

1.) The factor $\frac{1}{2!}$ arises because the r.h.s. of Eq. (17.7) has 12 terms of the type $\omega^j\otimes\omega^k$, not 6. The term with the common coefficient v^i occurs 2! times. This is because the unrestricted double sum $\epsilon_{ijk}\,\omega^j\wedge\omega^k=\epsilon_{ijk}\,\omega^j\otimes\omega^k-\epsilon_{ijk}\,\omega^k\wedge\omega^j=\epsilon_{ijk}\,\omega^j\otimes\omega^k+\epsilon_{ikj}\,\omega^k\otimes\omega^j=$ $=2\,\epsilon_{ijk}\,\omega^j\otimes\omega^k$. Because of this, one introduces the restricted double sum without the factor

$$\boxed{*\mathfrak{j}=N\,v^i\epsilon_{i|jk|}\,\omega^j\wedge\omega^k\equiv N\,v^i\sum_{j<k}\sum\epsilon_{ijk}\,\omega^j\wedge\omega^k}$$

2.) Physically the expression $*\mathfrak{j}$ is the number of particles per unit time passing through the (oriented) area spanned by a pair of as-yet-unspecified vectors;

$$*\mathfrak{j}=\text{"}\left[\frac{(\text{particles})}{(\text{time})(\text{area})}\right]\text{"}$$

3.) Mathematically one says that $*\mathfrak{j}$, an antisymmetric rank $\binom{0}{2}$ tensor,

$$*\mathfrak{j}:\;\; V\times V\longrightarrow R$$
$$(\vec{A}_1,\vec{A}_2)\rightsquigarrow *\mathfrak{j}(A_1,A_2)=N\,v^i\epsilon_{i|jk|}\,\omega^j\wedge\omega^k(\vec{A}_1,\vec{A}_2),$$

which is a scaler-valued two-form.

4.) The fact that Eq. (17.7) is a (basis independent) scalar implies that this scalar is an inner product of $\mathfrak{j}=N\,v^i e_i$, the particle flux vector, and another vector which is entirely independent of N and v^i. This vector is identified with the help of the inner products $g_{m\ell}=e_m\cdot e_\ell$ and their inverse matrix $g^{\ell i}$;

$$e_m \cdot e_\ell \, g^{\ell i} = \delta_m^{\ i}.$$

Introduce this unit matrix into Eq.(17.7)

$$*\mathcal{J}(\vec{A}_1, \vec{A}_2) = N v^m e_m \cdot e_\ell \, g^{\ell i} \, \epsilon_{ijk} \, \omega^j \wedge \omega^k / 2! \, (\vec{A}_1, \vec{A}_2)$$

$$= \mathcal{J} \cdot \underbrace{e_\ell \, g^{\ell i} \, \epsilon_{ijk} \, \omega^j \wedge \omega^k / 2! \, (\vec{A}_1, \vec{A}_2)}_{\equiv \; e_\ell \, d^2\Sigma^\ell}$$

and infer that the to-be-intified vector is

$$e_\ell \otimes d^2\Sigma^\ell(\vec{A}_1, \vec{A}_2) = e_\ell \otimes g^{\ell i} \, \epsilon_{ijk} \, \omega^j \wedge \omega^k / 2! \, (\vec{A}_1, \vec{A}_2).$$

This vector is clearly orthogonal to both $\vec{A}_1$ and $\vec{A}_2$. In fact, relative to an orthonormal frame it is the cross product

$$\boxed{\vec{A}_1 \times \vec{A}_2 = e_\ell \otimes d^2\Sigma^\ell (A_1, A_2).}$$ "cross product"

III. Generalizations to 4-d spacetime

1.) The generalization of the 3-d vector cross product to 4-d spacetime is

$$\boxed{e_\sigma \otimes d^3\Sigma^\sigma (A_1, A_2, A_3) = e_\sigma \otimes g^{\sigma\mu} \epsilon_{\mu\alpha\beta\gamma} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma / 3! \, (A_1, A_2, A_3)}$$

2.) The generalization of the 3-d "particle flux 2-form"

$$\boxed{*\mathcal{J} = N v^i \epsilon_{ijk} \, \omega^j \wedge \omega^k / 2!}$$

to the 4-d "particle density-flux 3-form" is

$$\boxed{*\mathbb{J} = N u^\mu \epsilon_{\mu\alpha\beta\gamma} \, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma / 3!}$$

3.) The generalization of the 3-d "particle flux vector"

$$\boxed{\mathcal{J} = N\vec{v} = N v^i e_i}$$

to the 4-d "particle density flux 4-vector" is

$$\boxed{\mathbb{J} = N\mathbf{u} = N u^\mu e_\mu}$$

where $\mathbf{u}$ is the common particle 4-velocity

1.) The factor $\frac{1}{2!}$ arises because the r.h.s. of Eq. (17.7) has 12 terms of the type $\omega^j \otimes \omega^k$, not 6. The term with the common coefficient v^i occurs 2! times. This is because the 1
unrestricted double sum $\epsilon_{ijk}\, \omega^j \wedge \omega^k = \epsilon_{ijk}\, \omega^j \otimes \omega^k - \epsilon_{ijk}\, \omega^k \wedge \omega^j = \epsilon_{ijk}\, \omega^j \otimes \omega^k + \epsilon_{ikj}\, \omega^k \otimes \omega^j =$
$= 2\, \epsilon_{ijk}\, \omega^j \otimes \omega^k$. Because of this one introduces the restricted double sum without the factor

1.) The factor $\frac{1}{2!}$ arises because both $\epsilon_{i23}\, \omega^2 \wedge \omega^3$ and $\epsilon_{i32}\, \omega^3 \wedge \omega^2$ and other equal terms like it are in the implied double sum $\sum_k \sum_l$. The factor $\frac{1}{2!}$ prevents excessive number of number of terms. Because of this one introduces the restricted sum without that factor,