LECTURE 19

I. The Particle Density-Flux 3-form

II. Density-flux coordinate representative of a uniform ensemble of particle trajectories.

Matter exists in different forms: solid, liquid, gas, plasma, super fluid and others. They have an identity which in any given inertial reference frame is conceptualized in terms of particle number, charge, energy, momentum and others. They are ruled by conservation laws.

These laws are universal, whatever the geometry, in particular the curvature of spacetime. It is by means of these laws that <u>curvature controls the motion of matter</u>, and it is via the Einstein field equations that <u>matter controls the curvature of spacetime.</u>

The mathematization of these laws is in terms of vectors and tensors on spacetime. They are 4-d generalizations of Faraday's and Maxwell's flux tube structures. The steps leading to such a generalization are illustrated when applied to an element of fluid containing a given number of particles.

Step I

Start with an element of fluid having uniform particle density N and 4-velocity u

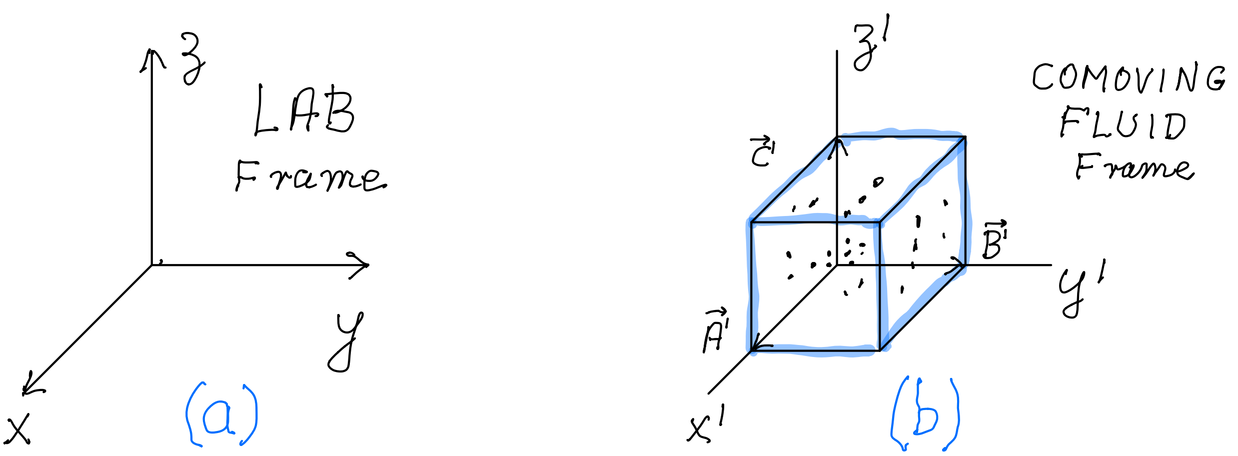

Figure 19.1: Element of fluid spanned by $(\vec{A}, \vec{B}, \vec{C})$ in panel (b) moves with velcity $\vec{u} = u^i e_i$ in the LAB frame.

Let $\{e_{1'}, e_{2'}, e_{3'}\}$ and $\{\omega^{1'}, \omega^{2'}, \omega^{3'}\}$ be an o.n. basis in the comoving fluid frame.

Let # = number of particles in the volume element spanned by the vectors $\vec{A}'$, $\vec{B}'$ and $\vec{C}'$ as determined for particles located at the origin and at $\vec{A}'$, $\vec{B}'$ and $\vec{C}'$.

This number is

$$\# = N \det \begin{vmatrix} \omega^{1'}(\vec{A}') & \omega^{2'}(\vec{A}') & \omega^{3'}(\vec{A}') \\ \omega^{1'}(\vec{B}') & \omega^{2'}(\vec{B}') & \omega^{3'}(\vec{B}') \\ \omega^{1'}(\vec{C}') & \omega^{2'}(\vec{C}') & \omega^{3'}(\vec{C}') \end{vmatrix} \equiv N\, \omega^{1'}\wedge\omega^{2'}\wedge\omega^{3'}(\vec{A}', \vec{B}', \vec{C}') \qquad (19.1)$$

Relative to a non-orthonormal basis $\{e_i\}$ and $\{\omega^j\}$ whose inner product matrix $[e_i \cdot e_j] = [{}^{(3)}g_{ij}]$ is not the unit matrix, that number is

$$\# = |{}^{(3)}g|^{1/2} N\, \omega^{1'}\wedge\omega^{2'}\wedge\omega^{3'}(\vec{A}', \vec{B}', \vec{C}'),$$

Here ${}^{(3)}g$ is the 3-d determinant of $[{}^{(3)}g_{ij}]$.

Step II

For the COMOVING FLUID frame introduce the Lorentz o.n. basis

$$\{e_{0'}, e_{1'}, e_{2'}, e_{3'}\} \equiv \{e_{\mu'}\} : \; e_{\mu'} \cdot e_{\nu'} = \eta_{\mu'\nu'} = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

and its dual basis

$$\{\omega^{0'}, \omega^{1'}, \omega^{2'}, \omega^{3'}\}.$$

Relative to this comoving Lorentz frame the particle number Eq. (19.1) in the fluid element is

$$\# = \det \begin{vmatrix} Nu^{0'} & Nu^{1'} & Nu^{2'} & Nu^{3'} \\ \omega^{0'}(A') & \omega^{1'}(A') & \omega^{2'}(A') & \omega^{3'}(A') \\ \omega^{0'}(B') & \omega^{1'}(B') & \omega^{2'}(B') & \omega^{3'}(B') \\ \omega^{0'}(C') & \omega^{1'}(C') & \omega^{2'}(C') & \omega^{3'}(C') \end{vmatrix} \qquad (19.2)$$

That Eqs. (19.2) and (19.1) are equal follows from the fact that (i) the 4-vectors A', B' and C' are purely space-like,

$$\omega^{0'}(A') = \omega^{0'}(B') = \omega^{0'}(C') = 0$$

$$\left.\begin{aligned} \omega^{i'}(A') &= \omega^{i'}(\vec{A}') \\ \omega^{i'}(B') &= \omega^{i'}(\vec{B}') \\ \omega^{i'}(C') &= \omega^{i'}(\vec{C}') \end{aligned}\right\} \quad i' = 1', 2', 3'$$

and (ii) that relative to the comoving frame the fluid 4-velocity has no space-like components:

$$\mathbb{u} = \underbrace{1}_{u^{0'}} e_{0'} + \underbrace{0}_{u^{1'}} \cdot e_{1'} + \underbrace{0}_{u^{2'}} \cdot e_{2'} + \underbrace{0}_{u^{3'}} \cdot e_{3'} \qquad (19.3)$$

These observations are condensed into the following

Definition ("Comoving basis")

Given a fluid each of whose particles has 4-velocity $\mathbb{u}$, a spacetime basis $\{e_{\mu'}\}_{\mu'=0}^{3}$ is said to be comoving with the fluid if the 4-velocity of the fluid particles has the form

$$\mathbb{u} = 1 e_{0'} + 0\, e_{1'} + 0\, e_{2'} + 0\, e_{3'},$$

i.e. has no spatial components. The particles are at rest in this comoving frame of the element of fluid.

Step III

Introduce the LAB frame basis (or any other basis),

$$\{e_0, e_1, e_2, e_3\} \equiv \{e_\mu\}_{\mu=0}^{3} : e_\mu \cdot e_\nu = g_{\mu\nu}$$

and it dual basis

$$\{\omega^0, \omega^1, \omega^2, \omega^3\} = \{\omega^\nu\}_{\nu=0}^{3} : \ \langle \omega^\nu | e_\mu \rangle = \delta^\nu_\mu$$

relative to which the particles of the element of fluid have non-zero spatial velocity,

$$\mathbf{u} = u^0 e_0 + u^1 e_1 + u^2 e_2 + u^3 e_3 \, .$$

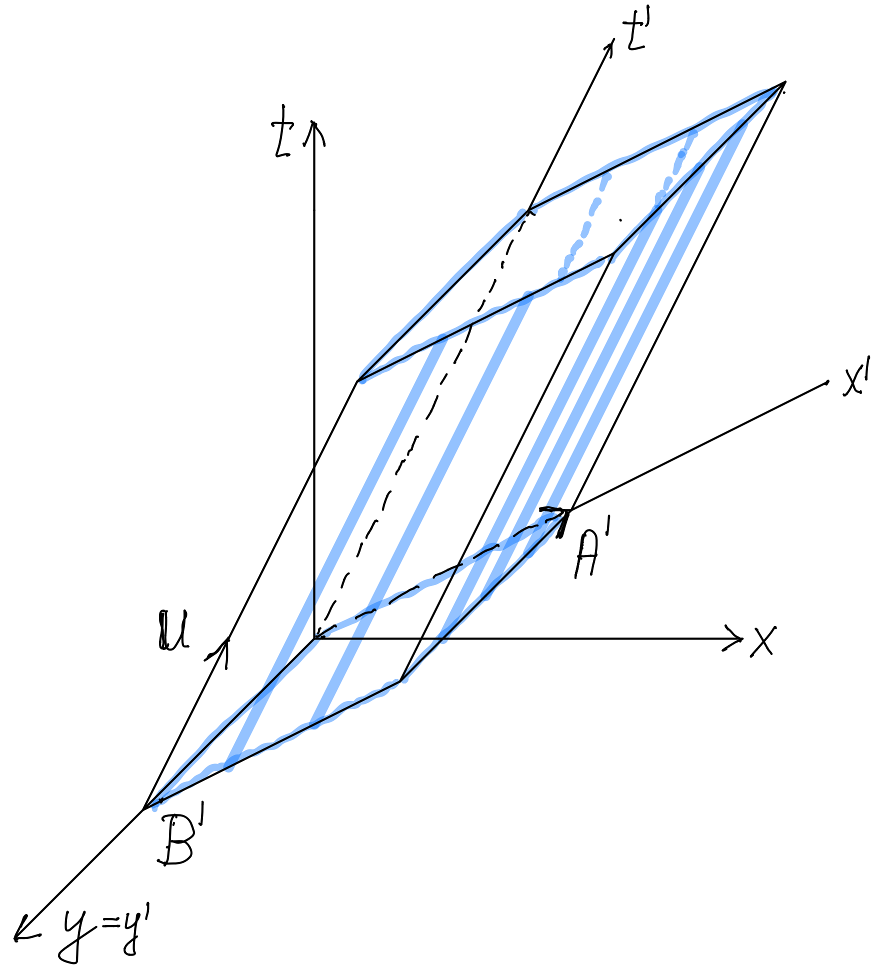

Figure 19.2: Spacetime history of a volume element of fluid particles with common 4-velocity $\mathbf{u}$.

The particle number # in the volume element is an invariant relative to any chosen basis:

$$\# = |g|^{1/2}\det\begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ \omega^0(A') & \omega^1(A') & \omega^2(A') & \omega^3(A') \\ \omega^0(B') & \omega^1(B') & \omega^2(B') & \omega^3(B') \\ \omega^0(C') & \omega^1(C') & \omega^2(C') & \omega^3(C') \end{vmatrix} \equiv \underbrace{\overbrace{Nu^\mu}^{J^\mu}\, \overbrace{\epsilon_{\mu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma/3!}^{d^3\Sigma_\mu}}_{*\mathbb{J}}\,(A', B', C')$$

This equation extends mathematically Faraday and Maxwell's flux tube concept

$$*\mathcal{J} \equiv j^i\epsilon_{ijk}\,\omega^j\wedge\omega^k/2! = Nv^i\epsilon_{ijk}\,\omega^j\wedge\omega^k/2! \equiv Nv^i\, d^2\Sigma_i$$

from its 3-d domain to the 4-d spacetime domain

$$*\mathbb{J} \equiv J^\mu\epsilon_{\mu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma/3! = Nu^\mu\epsilon_{\mu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma/3! \equiv Nu^\mu\, d^3\Sigma_\mu .$$

This is a scalar valued totally antisymmmtric rank$\binom{0}{3}$ tensor:

$$*\mathbb{J}: \quad \underset{(A,\,B,\,C)}{V\times V\times V} \longrightarrow \mathbb{R} \qquad\qquad (= \#\text{of particles})$$

where

$$*\mathbb{J}(A,B,C) = |g|^{1/2}\det\begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ \omega^0(A) & \omega^1(A) & \omega^2(A) & \omega^3(A) \\ \omega^0(B) & \omega^1(B) & \omega^2(B) & \omega^3(B) \\ \omega^0(C) & \omega^1(C) & \omega^2(C) & \omega^3(C) \end{vmatrix} = \#\text{ of particles} \qquad (19.4)$$

$*\mathbb{J}$ is a linear combination consisting of four independent terms,

$$*\mathbb{J} = J^0\epsilon_{0123}\,\omega^1\wedge\omega^2\wedge\omega^3 + J^1\underbrace{\epsilon_{1023}}_{-\epsilon_{0123}}\,\omega^0\wedge\omega^2\wedge\omega^3 + J^2\underbrace{\epsilon_{2031}}_{-\epsilon_{0123}}\,\omega^0\wedge\omega^3\wedge\omega^1 + J^3\underbrace{\epsilon_{3012}}_{-\epsilon_{0123}}\,\omega^0\wedge\omega^1\wedge\omega^2$$

$$*\mathbb{J} = \left(J^0\omega^1\wedge\omega^2\wedge\omega^3 - J^1\omega^0\wedge\omega^2\wedge\omega^3 - J^2\omega^0\wedge\omega^3\wedge\omega^1 - J^3\omega^0\wedge\omega^1\wedge\omega^2\right)\epsilon_{0123} \qquad (19.5)$$

$$\equiv J_{123}\,\omega^1\wedge\omega^2\wedge\omega^3 - J_{023}\,\omega^0\wedge\omega^2\wedge\omega^3 - J_{031}\,\omega^0\wedge\omega^3\wedge\omega^1 - J_{012}\,\omega^0\wedge\omega^1\wedge\omega^2$$

Their coefficients have well-defined physical meanings.

(i) Relative to the <u>LAB</u> basis let the triad of 4-vectors be (19.7)

$$(A, B, C) = (\Delta x\, e_1, \Delta y\, e_2, \Delta z\, e_3)$$

Subjecting them to Eq. (19.5) yields

$$*\mathbb{J}(A, B, C) = J_{123}\, \Delta x \Delta y \Delta z = \# \qquad (19.6)$$

Subjecting them to Eq. (19.4) yields

$$*\mathbb{J}(A, B, C) = |g|^{1/2} \det \begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ 0 & \Delta x & 0 & 0 \\ 0 & 0 & \Delta y & 0 \\ 0 & 0 & 0 & \Delta z \end{vmatrix} = \#$$

$$= |g|^{1/2} Nu^0 \Delta x \Delta y \Delta z = \# \qquad (19.7)$$

COMPARE Eqs. (19.6) with (19.7) and find that J_{123} is

$$\boxed{J_{123} = \frac{\#}{\Delta x \Delta y \Delta z} = |g|^{1/2} Nu^0 = |g|^{1/2} J^0}$$

which is the particle density in the LAB.

(ii) Relative to the <u>LAB</u> basis let one of the triad of 4-vectors be a timelike vector,

$$(A, B, C) = (\Delta t\, e_0, \Delta y\, e_2, \Delta z\, e_3)$$

Subjecting them to Eq. (19.5) yields

$$*\mathbb{J}(A, B, C) = -J_{023}\, \Delta t \Delta y \Delta z = \# \qquad (19.8)$$

Subjecting them to Eq. (19.4) yields

$$*\mathbb{J}(A,B,C)=|g|^{1/2}\det\begin{vmatrix} Nu^0 & Nu^1 & Nu^2 & Nu^3 \\ \Delta t & 0 & 0 & 0 \\ 0 & 0 & \Delta y & 0 \\ 0 & 0 & 0 & \Delta z \end{vmatrix}=\#$$

$$=|g|^{1/2}(-)Nu^1\Delta t\Delta y\Delta z=\# \qquad (19.9)$$

COMPARE Eqs. (19.8) with (19.9) and find that J_{023} is

$$\boxed{J_{023}=\frac{\#}{(\Delta y\Delta z)\Delta t}=|g|^{1/2}Nu^1=|g|^{1/2}J^1}$$

This is the x-directed particle flux.

The two boxed equations on page 19.7 and 19.8 and two like it are summarized by the statements that the components of the density-flux tensor $*\mathbb{J}=Nu^\mu\epsilon_{\mu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma$ are

$J_{\alpha\beta\gamma}$: $\dfrac{\#}{(\Delta x\Delta y\Delta z)}=$ LAB particle density

$\dfrac{\#}{(\Delta y\Delta z)\Delta t}=$ LAB particle flux into the x-direction

$\dfrac{\#}{(\Delta z\Delta x)\Delta t}=$ LAB particle flux into the y-direction

$\dfrac{\#}{(\Delta x\Delta y)\Delta t}=$ LAB particle flux into the z-direction

On the other hand, the components of the current

4-vector $\mathbb{J} = J^\mu e_\mu$ are

$$J^0 = \frac{\#}{(\Delta x \Delta y \Delta z)} = \text{LAB particle density}$$

$$J^x = \frac{\#}{(\Delta y \Delta z)\,\Delta t} = \text{LAB current into the x-direction}$$

$$J^y = \frac{\#}{(\Delta z \Delta x)\,\Delta t} = \text{LAB current into the y-direction}$$

$$J^z = \frac{\#}{(\Delta x \Delta y)\,\Delta t} = \text{LAB current into the z-direction}$$

II. Particle density-flux of a uniform ensemble of particle trajectories.

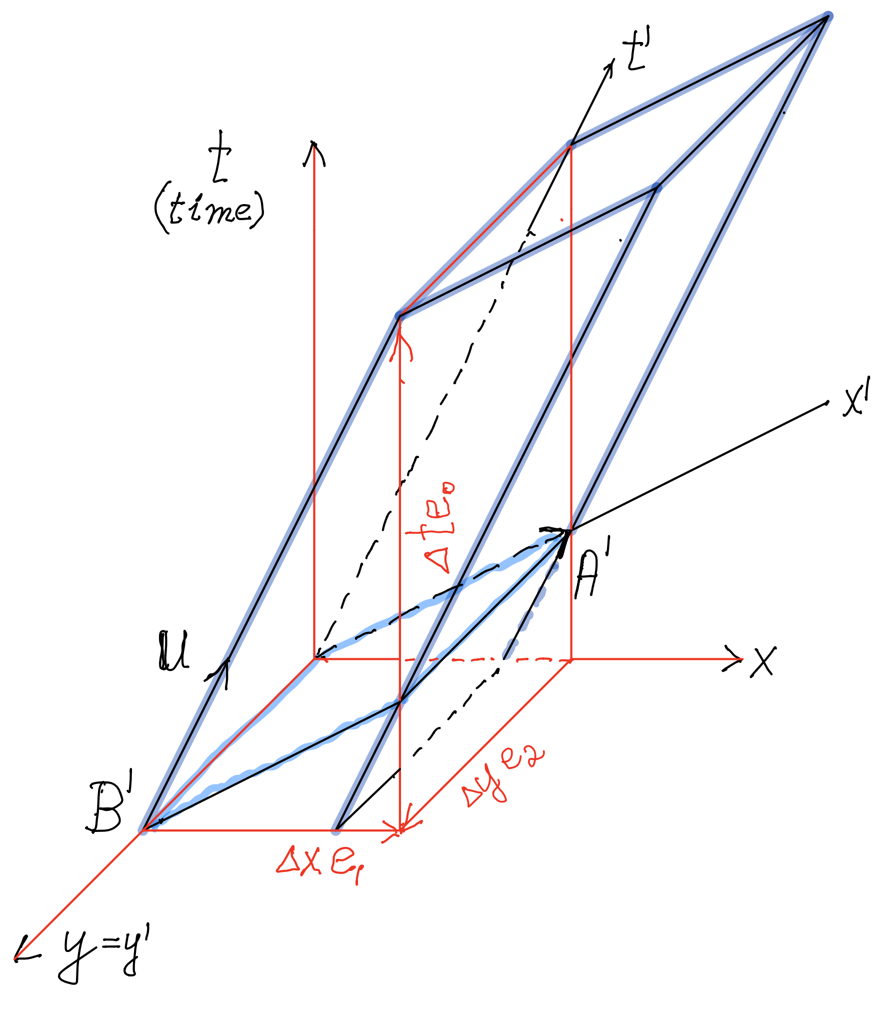

Figure 19.3 Space time diagram of the (blue) world tube of a volume element of fluid whose particles have, as depicted, the common 4-velocity u.

This volume element sweeps out the 4-d cube spanned by the 4-vectors u, A', B' and C'. The events A' and B' of the 4-vectors A' and B' are simultaneous relative to the reference frame comoving with that volume elements. These events (together with event C' not shown) form a triad of 4-vectors that span the 3-volume element $(\vec{A}', \vec{B}', \vec{C}')$ in Figure 19.1. This triad spans the initial (at $t'=0$) 3-d Lorentz orthogonal cross section of the (blue) 4-d world tube. The interior of this world tube is filled with its particle world lines (not shown in this figure). They all have the common 4-velocity u. This 4-velocity together with $A', B' C'$ span the 4-d world tube which is threaded in spagetti-like fashion by the unbroken particle world lines.

The LAB observer's spacetime domain is the 4-d cube spanned by the (red) 4-vectors $\Delta t\, e_0$, $\Delta x\, e_1$, $\Delta y\, e_2$, and $\Delta z\, e_3$ (not shown). Particles and their world lines passing through this spacetime are observed and counted by the LAB observer using triads of 4-vectors such as $(\Delta x\, e_1, \Delta y\, e_2, \Delta z\, e_3)$, or $(\Delta t\, e_0, \Delta x\, e_1, \Delta y\, e_2)$, $\cdots$, or $(\Delta t\, e_0, \Delta z\, e_3, \Delta x\, e_1)$.

By counting the number of particles in the space-like 3-volume of the triad $(\Delta x\, e_1, \Delta y\, e_2, \Delta z\, e_3)$ at $t=0$, the LAB observer arrives at the concept of

$$\text{"particle density"} = \frac{\#}{\Delta x \Delta y \Delta z} = \frac{(\text{particles})}{(\text{volume})}$$

By counting the number of particles in the time-like 3-volume, $(\Delta t\, e_0, \Delta y\, e_2, \Delta z\, e_3)$ at $x=\Delta x$, this observer arrives at the concept of

$$\text{"particle flux into the x-direction"} = \frac{\#}{\Delta t\, \Delta y\, \Delta z} = \frac{(\text{particles})}{(\text{time})\left(\begin{array}{c}\text{area directed into}\\ \text{the x-direction}\end{array}\right)}$$

Similarly one arrives at

$$\text{"particle flux into the y-direction"} = \frac{\#}{\Delta t\, \Delta z\, \Delta x} = \frac{(\text{particles})}{(\text{time})\left(\begin{array}{c}\text{area directed into}\\ \text{the y-direction}\end{array}\right)}$$

and

$$\text{"particle flux into the z-direction"} = \frac{\#}{\Delta t\, \Delta x\, \Delta y} = \frac{(\text{particles})}{(\text{time})\left(\begin{array}{c}\text{area directed into}\\ \text{the z-direction}\end{array}\right)}$$