LECTURE 2

Key Idea: Invariance of the Interval

T-W $\underbrace{3.6,\ 3.7,\ 3.8}_{\text{2nd edition}}$ $\underbrace{(1.5)}_{\text{1st edition}}$

LECTURE 2

2.2

The most important consequence of the Principle of Relativity is the Invariance of the Interval.

The "interval" between two point-events in spacetime is what in Euclidean space is the "distance" between two points. The "invariance" expresses the inertial frame independence of this interval.

In Euclidean space the distance between two points is independent of the relative orientation of the axes of different coordinate systems.

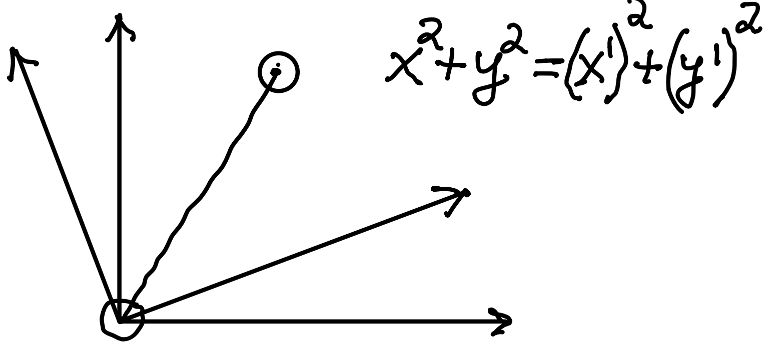


Figure 2.1:
Coordinate invariance of the distance between points in the Euclidean plane.

Similarly in spacetime the interval between two events recorded in different inertial frames is independent of their relative motion.

More precisely one has the following

Theorem (Invariance of the interval)

GIVEN: (a) two events in spacetime, $(t_1, \underset{\sim}{r}_1)$ and $(t_2, \underset{\sim}{r}_2)$

(b) Let two observers S and $\bar{S}$ measure the separation between these events in their respective inertial frames:

S measures $(\Delta t, \Delta \underset{\sim}{r})$ and

$\bar{S}$ measures $(\overline{\Delta t}, \overline{\Delta \underset{\sim}{r}})$

CONCLUSION:

$$\boxed{\text{The P. of R.} + \text{isotropy of space}} \Longrightarrow \boxed{(\Delta t)^2 - (\Delta \underset{\sim}{r})^2 = (\overline{\Delta t})^2 - (\overline{\Delta \underset{\sim}{r}})^2}$$

The proof is a 3 step process of deductive reasoning.

Comments

(i) The quantity $(\Delta t)^2 - (\Delta \underset{\sim}{r})^2 \equiv (\Delta \tau)^2$ is called the (squared) <u>interval</u> between events ① and ②.

(ii) The interval is an <u>invariant</u> because it is the same relative to all inertial frames.

(iii) We have defined time as distance traveled by light

$$\left.\begin{aligned} t &= c\, t_{\text{conv.}} \\ \bar{t} &= c\, \bar{t}_{\text{conv.}} \\ \bar{\bar{t}} &= c\, \bar{\bar{t}}_{\text{conv.}} \end{aligned}\right\} \quad c = 3\times 10^{10}\ \text{cm/sec} = 3\times 10^{8}\ \frac{\text{m}}{\text{sec}}$$

because c, the measured speed of light, is the same in all inertial frames.

For example:

$\Delta t = 1$ meter corresponds to

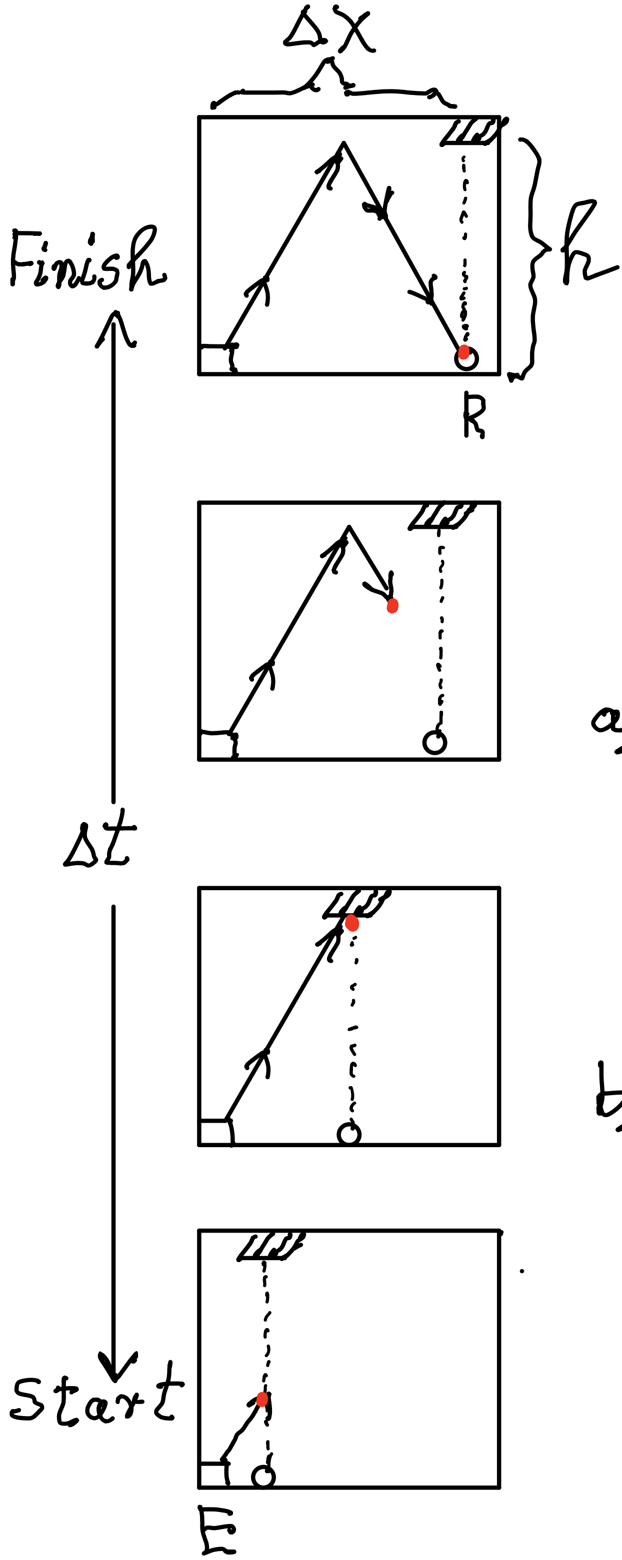
$\Delta t_{conv} = 3 \times 10^{-9}$ sec $= 3$ nanoseconds

because $c\,\Delta t_{conv} = 3 \times 10^{8} \frac{m}{sec} \times 3 \times 10^{-9}$ sec $= 1$ meter.

(End of Comments)

The deductive line of reasoning consists of first following the trajectory of a pulse of light as measured in the LAB frame and in a ROCKET frame moving relative to the LAB, then of expressing the longitudinal measurements in terms of those transverse to the relative motion, and finally of arriving the invariance from the P. of R.

(2.7)



Step I:

Elapsed time between emission event E and reception event R

a) as determined in the LAB frame

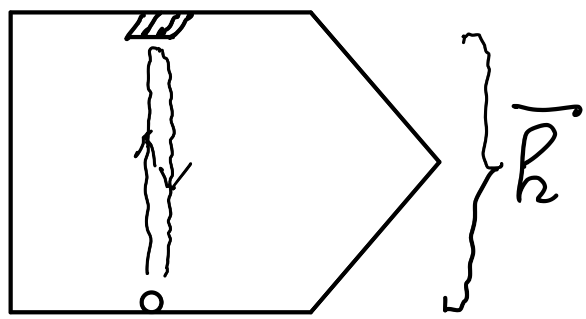
$$\Delta t = \Delta t_{conv} c = 2\sqrt{\left(\frac{\Delta x}{2}\right)^2 + h^2}$$

$$\Delta x = \Delta x$$

b) as determined in the ROCKET frame:

$$\overline{\Delta t} = \Delta \bar{t}_{conv} c = 2\bar{h}$$

$$\overline{\Delta x} = 0$$

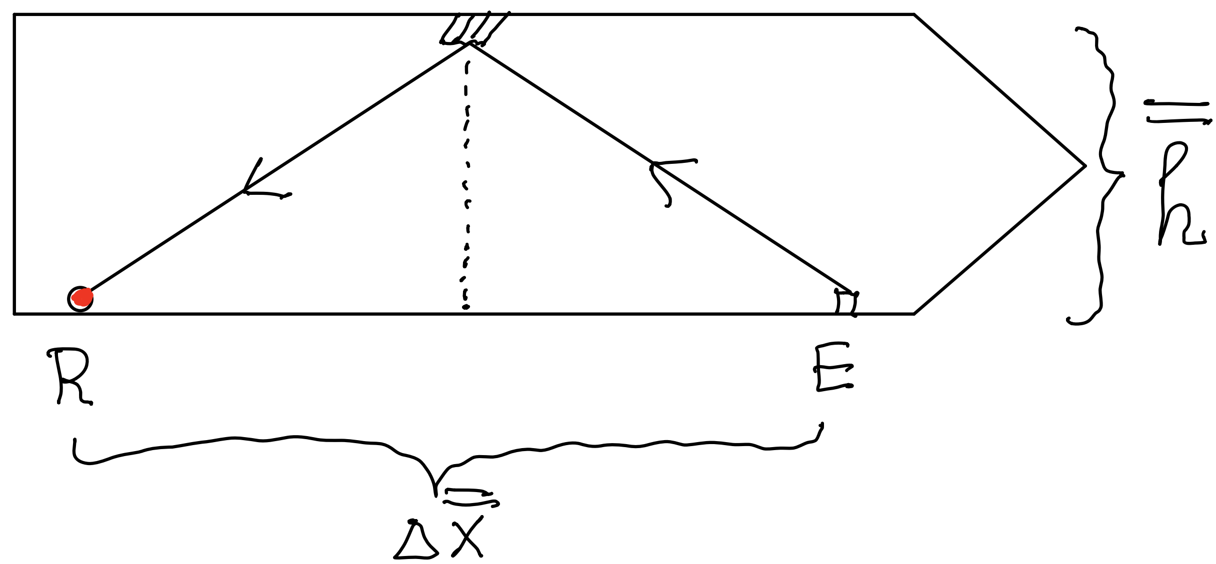


Rocket mirror (= ///) and clock (= ∘) moving in the lab frame (= □)

c) as determined in the SUPER ROCKET frame: (2.8)

$$\overline{\overline{\Delta t}} = \Delta \overline{\overline{t}}_{conv}\, c = 2\sqrt{\left(\frac{\overline{\overline{\Delta x}}}{2}\right)^2 + \overline{\overline{h}}^2}$$

$$\overline{\overline{\Delta x}} = \overline{\overline{\Delta X}}$$



Comment:

We have used

(i) the <u>isotropy</u> of space in a) and c)

(ii) the Principle of Relativity:

the law of light propagation, in particular the speed of light, is the <u>same</u> in <u>all</u> inertial frames.

Step II:

Upon squaring and subtracting one finds:

$$(\Delta t)^2 - (\Delta x)^2 = h^2 \quad \text{in the LAB frame}$$

$$(\overline{\Delta t})^2 - (\overline{\Delta x})^2 = \overline{h}^2 \quad \text{in the ROCKET frame}$$

$$(\overline{\overline{\Delta t}})^2 - (\overline{\overline{\Delta x}})^2 = \overline{\overline{h}}^2 \quad \text{in the SUPER ROCKET frame}$$

Step III:

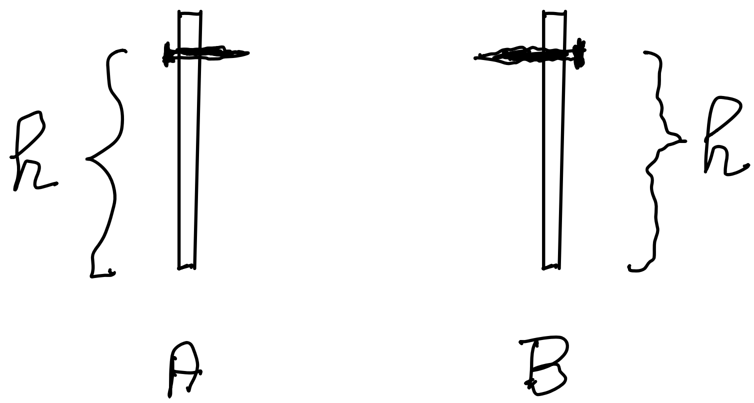
According to the Principle of Relativity

$$h = \overline{h}$$

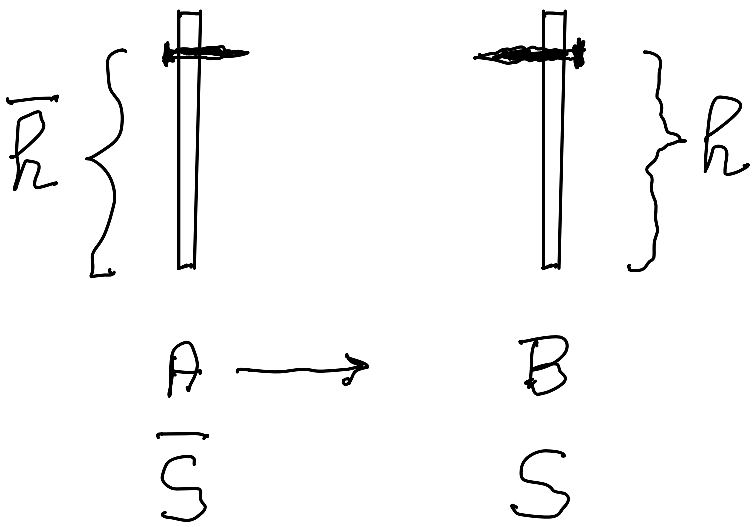
i.e. the length is the direction transverse to the direction of motion is the same in both frames: it is an <u>invariant</u>.

Q: Why so?

A: Start with two wooden boards, each with a nail at height h in it



Place board A into the ROCKET ($=\bar{S}$) frame and board B into the LAB ($=S$) frame



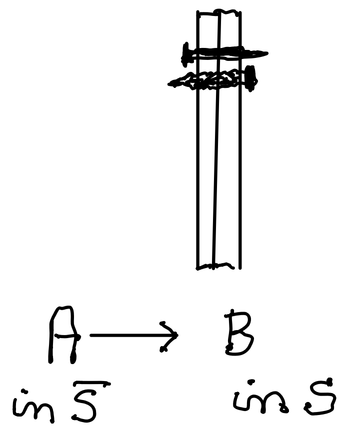
The LAB ($=S$) observer observes $\bar{S}$ approach from the left.

A and B collide. Applying the Principle of Relativity to the outcome of this process leads to the conclusion that

$$\bar{h} = h.$$

Indeed, assume that the contrary, say

$$\bar{h} > h.$$



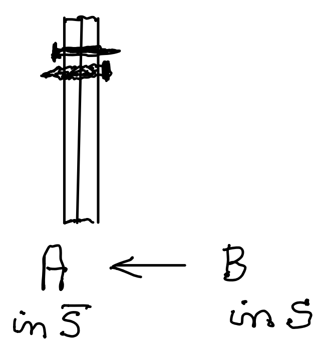
(WRONG)

S observes that in his own inertial frame S

A's nail leaves a mark above

B's nail in board B.

On the other hand one notes that

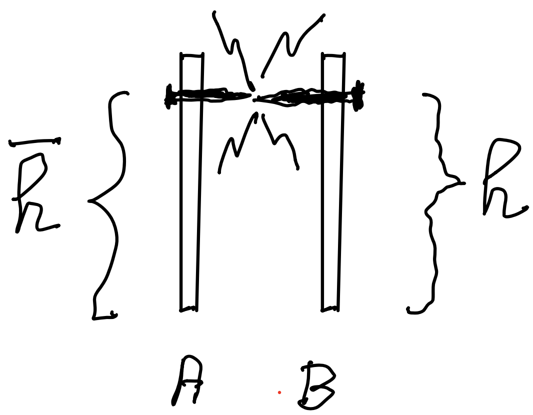


(WRONG)

$\bar{S}$ observes that in his own inertial frame $\bar{S}$
B's nail leaves a mark below
A's nail in board A.

This above-below difference between
S and $\bar{S}$ violates the P. of R.
Consequently, the collision between
A and B will result in sparks, i.e.

$$\bar{h} = h$$



Conclusion:

$$\boxed{(\Delta t)^2 - (\Delta x)^2 = (\Delta \bar{t})^2 - (\Delta \bar{x})^2 \equiv (\text{interval})^2}$$

More generally:

$$\boxed{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\overline{\Delta t})^2 - (\overline{\Delta x})^2 - (\overline{\Delta y})^2 - (\overline{\Delta z})^2 \equiv (\text{interval})^2}$$

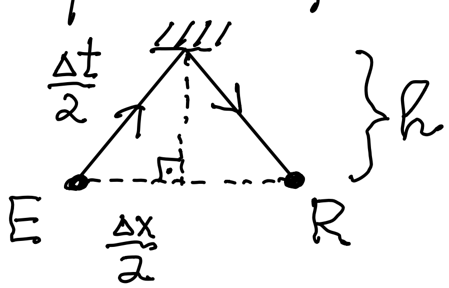
i.e. the interval between a pair of events is invariant; it is the same relative to all inertial reference frames

Summary

$$\boxed{\text{P. of R} + \begin{pmatrix}\text{isotropy}\\ \text{of space}\end{pmatrix} \Longrightarrow \begin{pmatrix}\text{invariance}\\ \text{of the}\\ \text{interval}\end{pmatrix}}$$

Comment

For the same pair of events E and R



the figures on Page 2.7 capture 4 central ideas:

1. $$(\text{hypotenuse})^2 - (\text{base})^2 = (\text{height})^2$$

$$\left(\tfrac{1}{2}\text{ time sep'n}\right)^2 - \left(\tfrac{1}{2}\text{ space sep'n}\right)^2 = \left(\tfrac{1}{2}\text{ interval}\right)^2$$

2. Invariance of the interval

3. Invariance of the speed of light

4. $\Delta t \neq \Delta \bar{t} \neq \Delta \bar{\bar{t}}$

$\Delta x \neq \overline{\Delta x} \neq \overline{\overline{\Delta x}}$