LECTURE 22

I. Tangent Vectors: Overview

II. Tangent Vector: Its Definition

III. Tangent Vector as a Coordinate Invariant Concept

Singer and Thorpe: Chapter 5: p 99-100

Comment: In their definition of a tangent vector on page 99 their wording is in terms of "then there exists an n-tuple $(a_1, a_2, \ldots, a_n)$ of..." The red underlined "exists" is an unfortunate choice of bad terminology. With a reader's background in linear algebra, this term suggests that the existence of that n-tuple is a matter of proving the existence and uniqueness of a system of n linear equation its n unknowns $a_1, a_2, \ldots, a_n$. This, however, is not all the case. Instead, these a_i's come about as the components of the tangent (velocity) of a given curve. In fact, Singer and Thorpe say/admit precisely that at the top of their page 104 in the context of their Figure 5.3.

I. Tangent vectors: Overview

A tangent vector is a mathematical method for geometrizing the concept of change in terms of a directional derivative so as to guarantee that it reflects the physical requirement of being invariant under ("independent of") one's choice of coordinates.

Given an n-dimensional manifold M, the coordinate charts of M accommodate the existence of a vector space

$$V = T_P(M),$$

the tangent space $T_P(M)$ at each point P of M. Each element of $T_P(M)$ is a tangent vector, an adaptation of the familiar direction derivative to the landscape (i.e. the coordinate charts) surrounding $P \in M$.

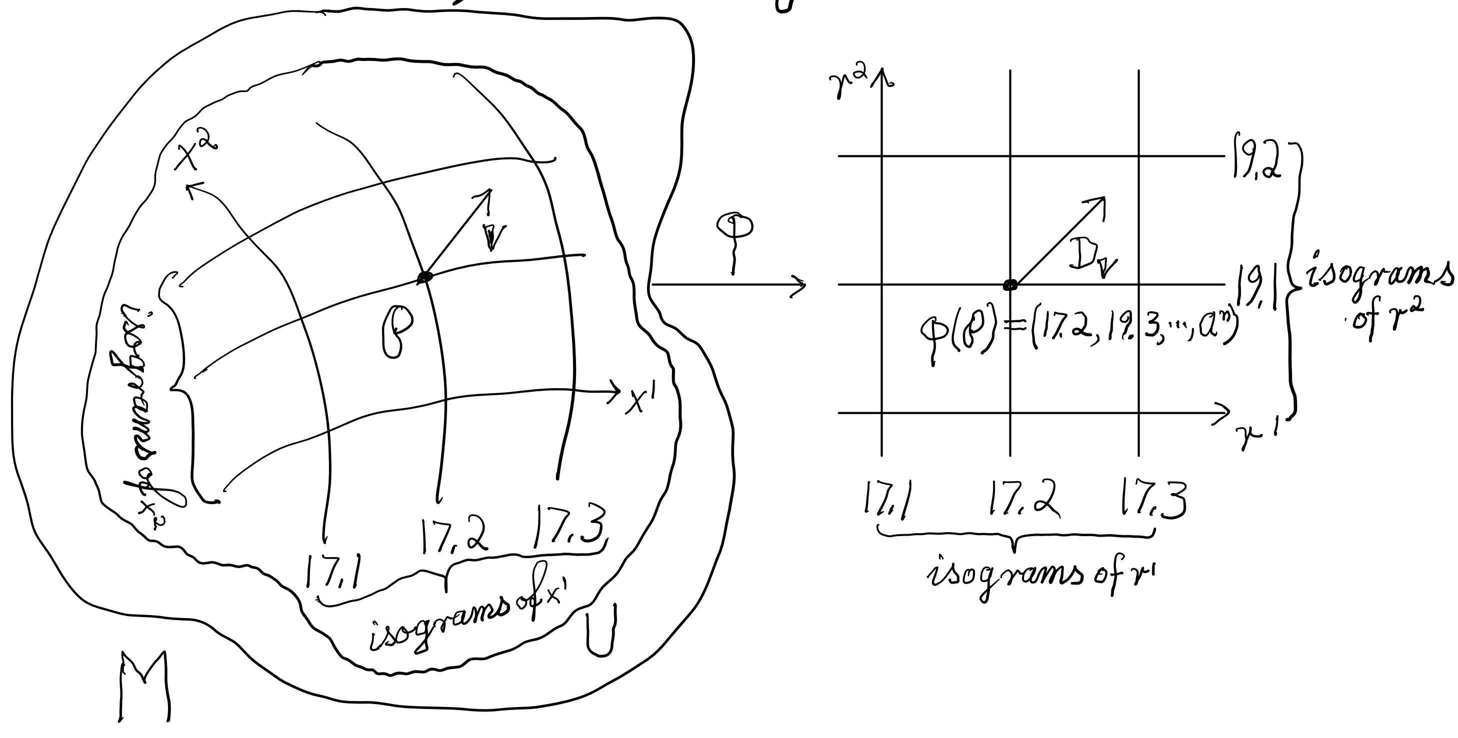

Figure 22.1: Vector V coordinatized as the directional derivative D_V relative to the coordinate chart (φ, U). The point P is coordinatized by the components $(\varphi(P))^i \equiv x^i(P) \equiv r^i \circ \varphi(P)$, $i = 1, \dots n$. The x^i's are called the coordinate functions on M, while the r^i's are the coordinate functions on R^n.

Thus,

(i) a tangent vector at a point is a directional derivative that sends all functions smooth around that point into the reals, and
(ii) that tangent vector is determined by its values on all smooth functions.

Properties (i) and (ii) serve as the basis for defining a tangent vector at $P \in M$.

II. Tangent Vector: Its Definition

Definition (Tangent vector as a derivation)

Let $P \in M$ be a point in the manifold M.
A tangent vector V at P is the map

$$
\begin{array}{llll}
V: & C^\infty(M, P, R) & \longrightarrow & R \\
 & \left(\begin{array}{c}\text{smooth real} \\ \text{functions in a} \\ \text{nbhd of } P \in M\end{array}\right) & \longrightarrow & \text{Reals} \\
 & f & \rightsquigarrow & V(f) \in R
\end{array}
$$

which, relative to a chosen coordinate chart (φ_U, U) and for a given n-tuple $(\alpha^1, \dots, \alpha^n)$, is, at point P, the $(\alpha^1, \dots, \alpha^n)$-determined directional derivative of f:

$$V(f)=\sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i}\, f\circ\varphi_U^{-1}(r^1,\dots,r^n)\Big|_{\varphi(P)=(a^1,\dots,a^n)} \qquad (22.1a)$$

$$\sum_{i=1}^{n} \alpha^i\, D_i\, f_{U\text{-rep}}(r^1,\dots,r^n)\Big|_{(a^1,\dots,a^n)} \qquad (22.1b)$$

This is the $\{\alpha^i\}$ linear combination of the partial derivatives of $f_{U\text{-rep}}$ at the point $(a^1, \dots, a^n)$, the coordinate image in R^n of point P. Thus there is a one-to-one correspondence between V and the given n-tuple $(\alpha^1, \dots, \alpha^n)$:

$$\boxed{V \leftrightsquigarrow (\alpha^1,\dots,\alpha^n)\in R^n}\,.$$

However, this definition lacks objectivity [lookup "objectivity" in the Ayn Rand Lexicon]. This is because it is subjective: it depends solely on the observer: on the chosen coordinate chart. In order to get reality right, one must show that this definition must hold also for those charts which have a non-zero intersection with (φ_U, U) given in the definition.

III. Invariance under coordinate transformation

The tangent vector $\mathbb{V} : C^\infty(M,R,\mathcal{P}) \longrightarrow R^1$ at point $\mathcal{P}$ has as its defining property the formula

$$\mathbb{V}(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f \circ \varphi_U^{-1}(r^1,\dots,r^n)$$

$$= \sum_{i=1} \alpha^i \, D_i \, f_{U\text{-rep}}(r^1,\dots,r^n)$$

relative to chart (φ_U, U).

Relative to an overlapping chart (ψ_V, V) use the transition map

$$\bar{r}^j(r^1,\dots,r^n) = \left(\psi_V \circ \varphi_U^{-1}(r^1,\dots,r^n)\right)^j = \bar{r}^j \circ \psi_V \circ \varphi_U^{-1}(r^1,\dots,r^n)$$

to obtain

$$\mathbb{V}(f)\Big|_{\mathcal{P}} = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f \circ \varphi_U^{-1}(r^1,\dots,r^n)\Big|_{\varphi_U(\mathcal{P})}$$

$$= \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} \underbrace{f \circ \psi_V^{-1}}_{f_{V\text{-rep}}(\bar{r}^1,\dots,\bar{r}^n)} \circ \underbrace{\psi_V \circ \varphi_U^{-1}(r^1,\dots,r^n)}_{\{\bar{r}^j(r^1,\dots,r^n)\}}\Big|_{\varphi_U(\mathcal{P})}$$

The partial derivative $\frac{\partial}{\partial r^i}$ are those of a composite function with intermediate variables $\bar{r}^1,\dots,\bar{r}^n$. Explicitly one has

$$\mathbb{V}(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f_{V\text{-rep}}\left(\bar{r}^1(r^1,\dots,r^n),\dots,\bar{r}^n(r^1,\dots,r^n)\right).$$

Use the chain rule to obtain

$$\mathbf{v}(f)=\sum_{i=1}^{n}\alpha^i\sum_{j=1}^{n}\frac{\partial\bar{r}^j}{\partial r^i}\bigg|_{\varphi_U(\mathcal{P})}\frac{\partial}{\partial\bar{r}^j}f_{V\text{-rep}}(\bar{r}^1,\dots,\bar{r}^n)\bigg|_{\psi_V(\mathcal{P})}$$

Here the partial derivatives

$$\frac{\partial\bar{r}^j(r^1,\dots,r^n)}{\partial r^i}=\frac{\partial}{\partial r^i}\,\bar{r}^j\circ\psi_V\circ\varphi_U^{-1}(r^1,\dots,r^n) \qquad (22.2)$$

are the elements of the <u>Jacobian matrix</u>

$$\left[\frac{\partial\bar{r}^j}{\partial r^i}\right]\equiv\left[J^j{}_i(\psi_V\circ\varphi_U^{-1})\right] \qquad (\text{"Jacobian"})$$

It transforms the φ_U coordinate components $\{\alpha^i\}$ to

$$\boxed{\sum_{i=1}^{n}\alpha^i\frac{\partial\bar{r}^j}{\partial r^i}\equiv\beta^j} \qquad j=1,\dots,n,$$

of the new ψ_V coordinate system. Relative this system the defining relation for $\mathbf{v}$ has the form

$$\mathbf{v}(f)=\sum_{j=1}^{n}\beta^j\,\frac{\partial f_{V\text{-rep}}(\bar{r}^1,\dots,\bar{r}^n)}{\partial\bar{r}^j}\bigg|_{\psi_V(\mathcal{P})}.$$

$$=\sum_{j=1}\beta^j\,\frac{\partial}{\partial\bar{r}^j}f\circ\psi_V^{-1}\bigg|_{\psi_V(\mathcal{P})}, \qquad (22.3)$$

which is the same relative to (φ_U, U),

$$\mathbf{v}(f)=\sum_{i=1}^{n}\alpha^i\,\frac{\partial}{\partial r^i}f\circ\varphi_U^{-1}\bigg|_{\varphi_U(\mathcal{P})}. \qquad (22.4)$$

The principle of unit-economy [see "unit-economy" in the Ayn Rand Lexicon] demands that this form equivalence be reflected in the notation for the vector $\mathbb{v}$. This is achieved by recalling from Figure 22.1 that on the neighborhood of the coordinate chart (φ_U, U) its components are the coordinate functions

$$x^i: \quad U \longrightarrow R$$
$$P \rightsquigarrow x^i(P) \equiv r^i \circ \varphi_U(P) \equiv (\varphi_U(P))^i.$$

In light of this, the unit-economical notation for Eq. (22.3) is

$$\frac{\partial(f)}{\partial x^i} \equiv \frac{\partial}{\partial r^i}\left(f \circ \varphi_U^{-1}\right)\Big|_{\varphi_U(P)} = \frac{\partial}{\partial r^i} f_{U\text{-rep}}(r^1, \dots, r^n)$$

for $f \in C^\infty(M, P, R)$. Thus, $\frac{\partial}{\partial x^i}$ corresponds relative to the coordinate system φ_U to the n-tuple $(0, \dots, 1, \dots, 0)$, where its i^{th} entry is 1.

More generally one has

$$\mathbb{v}(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial x^i}(f),$$

or by leaving f as-yet-unspecified,

$$\boxed{\mathbb{v} = \alpha^i \frac{\partial}{\partial x^i}} \qquad (22.5)$$

Thus $\mathbb{v}$ is the rate of change of an as-yet-unspecified function into the direction of the n-tuple $(\alpha^1, \dots, \alpha^n)$ relative to the coordinates φ.

In a similar way, based on the coordinate functions y^j, one obtains

$$\frac{\partial(f)}{\partial y^j} \equiv \frac{\partial}{\partial \bar{r}^j}\left(f \circ \psi_V^{-1}\right)\Big|_{\psi_V(P)} = \frac{\partial}{\partial \bar{r}^j} f_{V\text{-rep}}(\bar{r}^1, \dots, \bar{r}^n)$$

and

$$\boxed{V = \beta^j \frac{\partial}{\partial y^j}} \qquad (22.6)$$

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Combining Eqs. (22.5)-(22.6) obtain

$$\alpha^i \frac{\partial}{\partial x^i} = V = \beta^j \frac{\partial}{\partial y^j},$$

which is to say that a vector at a point is a coordinate independent object.

If x^i and y^j are the coordinate functions on $U \cap V$, then reference to Eq. (22.2) yields

$$\frac{\partial}{\partial x^i} = \frac{\partial(y^j)}{\partial x^i} \frac{\partial}{\partial y^j}.$$

where $\frac{\partial(y^j)}{\partial x^i}$ are the elements of the Jacobian matrix.