

LECTURE 23

23.1

- I. Tangent Vector
- II. Tangent Vector as a derivation
- III. The Tangent Space $T_p(M)$
- IV. The Cotangent Space $T_p^*(M)$
- V. Coordinate Transformation

Read Singer & Thorpe's
"Lecture Notes on Elementary Geometry and Topology",
pages 100-101

I. Tangent Vector

The concept of a tangent vector is the keystone by which one grasps multi-variable calculus and linear algebra from a single perspective. It is the the portal from linear to non-linear mathematics.

It maps C^∞ scalar functions on a manifold neighborhood U surrounding a point P into the reals

$$v: C^\infty(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

$$f \longmapsto V(f) \equiv \alpha^i \frac{\partial}{\partial x^i} (f \circ \varphi_u^{-1}) \Big|_{\varphi_u(P)}$$

$$= \alpha^i \frac{\partial}{\partial x^i} f_{u\text{-rep}}(x^1, \dots, x^n) \Big|_{\varphi_u(P)}$$

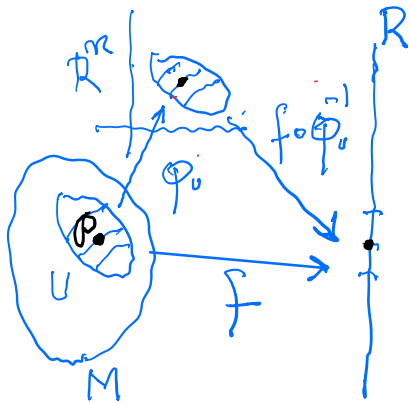


Figure 23.1: Scalar function f as a superstructure rooted in a base-manifold M , the foundation for all geometrical superstructures.

If $x^j = r^j \circ \varphi_u^{-1}$, $j=1, \dots, n$ are φ_u 's coordinate functions, then by the principle of unit-economy [according to which one reduces a vast amount of information (here, that of the manifold substructure) to a minimal number of units] one writes

$$v(f) = \alpha^i \frac{\partial f}{\partial x^i} \quad (23.1)$$

for all functions smooth at P . Thus one arrives at the conclusion that

$$v = \alpha^i \frac{\partial}{\partial x^i} \quad (23.2)$$

Comment.

1.) This is a mapping that sends smooth functions into the values of their

directional derivative $D_v = \alpha^i \frac{\partial}{\partial x^i}$ evaluated at the point P .

2.) The distinguishing feature of the set of tangent vectors v at P is their one-to-one correspondence with n -tuples

$$v \longleftrightarrow (\alpha^1, \dots, \alpha^n) \in \mathbb{R}^n$$

at P relative to the coordinate chart (φ_0, U) .

II. Tangent Vector as the Derivation at a Point.

The concept "tangent vector v " subsumes and integrates a non-trivial multiplicity of diverse components, attributes, and environments. Nevertheless, all of them can be condensed into a triad of defining statements:

For any $f, g \in C^\infty(M, \mathbb{R})$ and for $\lambda \in \mathbb{R}$

$$(1) \quad v(f+g) = v(f) + v(g)$$

$$(2) \quad v(\lambda f) = \lambda v(f)$$

$$(3) \quad v(fg) = f v(g) + v(f) g$$

"linearity"

"the product rule"

A mapping which satisfies these three properties is called a derivation. In addition, starting with a mapping having these properties, one can show that relative to any coordinate system, say (φ_0, U) containing P , there exist $(\alpha^1, \dots, \alpha^n) \in \mathbb{R}^n$

such that

$$v = \alpha^i \frac{\partial}{\partial x^i}$$

where x^i is the i^{th} coordinate function of φ_0 .

Conclusion

A tangent vector is a derivation, and a derivation is a tangent vector.

III. The Tangent Space $T_p(M)$ at $p \in M$

23.4

a) The fact that numbers can be added and multiplied by scalars so as to result in new numbers can be extended to the set $T_p(M)$ of tangent vectors at point p in M .

In that case one says that the set $T_p(M)$ of tangent vectors at p is a vector space: it is closed under addition and scalar multiplication of its elements:

$$(i) (V_1 + V_2)(f) = V_1(f) + V_2(f) \quad V_1, V_2 \in T_p(M); \quad \forall f \in C^\infty(M, \mathbb{R})$$

$$(ii) (\lambda V)(f) = \lambda V(f) \quad V \in T_p(M); \quad \forall f \in C^\infty(M, \mathbb{R})$$

b) The vector space $T_p(M)$ is isomorphic to \mathbb{R}^n . Indeed,

$$\mathbb{R}^n \longleftrightarrow T_p(M)$$
$$(\alpha^1, \dots, \alpha^n) \longleftrightarrow \alpha^i \frac{\partial}{\partial x^i}$$

is an isomorphism induced by the coordinate chart (φ_ν, ν) .

c) The chart (φ_ν, ν) causes $T_p(M)$ to have the natural coordinate basis

$$\left\{ \delta_1^i \frac{\partial}{\partial x^i}, \dots, \delta_n^i \frac{\partial}{\partial x^i} \right\} = \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\} = \{ e_1, \dots, e_n \}$$

Thus

$$T_p(M) = \text{span} \left(\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^n \right)$$

d) The transition from one coordinate chart to another induces a corresponding change in bases from $\left\{ \frac{\partial}{\partial x^i} \right\}$ to $\left\{ \frac{\partial}{\partial y^j} \right\}$. They are related by

$$\frac{\partial}{\partial x^i} = \frac{\partial(y^j)}{\partial x^i} \frac{\partial}{\partial y^j},$$

where

$$\frac{\partial(y^j)}{\partial x^i} = J_i^j(\psi_\nu \circ \varphi_\nu^{-1}) = \left. \frac{\partial T^j}{\partial x^i}(x^1, \dots, x^n) \right|_{\varphi(p)}$$

are the matrix elements of the Jacobian matrix between the two coordinate systems (φ, U) and (ψ, V) .

23.5

Example ("Trivial Tangent Space")

Consider $M = \mathbb{R}^n$ and $P = a \in \mathbb{R}^n$. Then the tangent space $T_a(\mathbb{R}^n)$ at the point $a \in \mathbb{R}^n$ is naturally isomorphic to \mathbb{R}^n itself. The isomorphism

$$\mathbb{R}^n \longleftrightarrow T_a(\mathbb{R}^n)$$

is given by

$$(\lambda^1, \dots, \lambda^n) \longmapsto \lambda^i \frac{\partial}{\partial x^i}.$$

IV. The Cotangent Space $T_p^*(M)$ dual to $T_p(M)$.

As a result of Eqs. (23.1)-(23.2), each function f in the domain $C^\infty(M, \mathbb{R})$ of

$$V: f \mapsto V(f)|_p = \alpha^i \frac{\partial f}{\partial x^i}$$

determines a unique linear function on $T_p(M)$. It is given by the Definition ("The differential of f ")

$$df: T_p(M) \longrightarrow \mathbb{R}$$

$$V \longmapsto \langle df | V \rangle|_p = V(f)|_p$$

$$\longmapsto \langle df | \alpha^i \frac{\partial}{\partial x^i} \rangle = \alpha^i \frac{\partial f}{\partial x^i}$$

By virtue of this definition df is linear on $T_p(M)$. This is because

$$\begin{aligned} (i) \quad \langle df | V_1 + V_2 \rangle &= (V_1 + V_2)(f) \\ &= V_1(f) + V_2(f) \\ &= \langle df | V_1 \rangle + \langle df | V_2 \rangle \end{aligned}$$

and

$$\begin{aligned} (ii) \quad \langle df | \lambda V \rangle &= (\lambda V)(f) \\ &= \lambda V(f) \\ &= \lambda \langle df | V \rangle. \end{aligned}$$

The linear function df on $T_p(M)$ is called the differential (or the differential form) of f at the point P .

a) The set of differential forms at point P , $T_p^*(M)$, is a vector space. Indeed, it is closed under addition and scalar multiplication:

$$\begin{aligned} \text{(i)} \quad \langle df+dg|V \rangle &\equiv \langle df|V \rangle + \langle dg|V \rangle \\ &= v(f) + v(g) \quad \left(= \alpha^i \frac{\partial f \circ \varphi^{-1}}{\partial r^i}(r^1, \dots, r^n) + \alpha^i \frac{\partial g \circ \varphi^{-1}}{\partial r^i}(r^1, \dots, r^n) \right) \\ &= v(f+g) \\ &= \langle d(f+g)|V \rangle. \end{aligned}$$

Thus $df+dg$ is the differential of $f+g$ at P and therefore belongs to $T_p^*(M)$. Similarly, one has

$$\begin{aligned} \text{(ii)} \quad \langle \lambda df|V \rangle &\equiv \lambda \langle df|V \rangle \\ &= \lambda v(f) \\ &= v(\lambda f) \\ &= \langle d(\lambda f)|V \rangle \end{aligned}$$

Thus $\lambda df = d(\lambda f)$ is the differential of λf at P .

Consequently, $T_p^*(M)$ is a vector space indeed.

b) Every given coordinate system induces its natural coordinate basis for $T_p(M)$. The corresponding dual basis is given by the following

Proposition ("Basis Dual to a Coordinate Basis")

Let $\{e_i \equiv \frac{\partial}{\partial x^i}\}_{i=1}^n$ be the natural coordinate basis for $T_p(M)$ induced by (φ, U) . 23.7

Let $\{x^j\}_{j=1}^n$ be the coordinate functions of φ .

Conclusion:

$$\langle dx^j | \frac{\partial}{\partial x^i} \rangle = \delta^j_i.$$

i.e. $\{dx^j\}$ is the basis dual to $\{e_i \equiv \frac{\partial}{\partial x^i}\}$ at p .

By letting in turn

$$V = \frac{\partial}{\partial x^i} \quad i=1, \dots, n$$

$$f = x^j \quad j=1, \dots, n$$

and computing $V(f)$ one finds using $\langle df | V \rangle = V(f) = \alpha^i \frac{\partial f \circ \varphi(r^1, \dots, r^n)}{\partial r^i}$ that

$$\langle dx^j | \frac{\partial}{\partial x^i} \rangle = \frac{\partial x^j \circ \varphi(r^1, \dots, r^n)}{\partial r^i}$$

$$\parallel = \frac{\partial x^j}{\partial r^i}$$

$$\langle dx^j | \frac{\partial}{\partial x^i} \rangle = \delta^j_i.$$

One says that $\{dx^j\}$ is the natural coordinate basis for $T_p^*(M)$ induced by (φ, U) .

Thus

$$T_p^*(M) = \text{span}(\{dx^j\}_{j=1}^n)$$

This to be compared and contrasted with

$$T_p(M) = \text{span}(\{\frac{\partial}{\partial x^i}\}_{i=1}^n)$$

and

$$\left\{ \frac{\partial}{\partial x^i} \right\} \longleftrightarrow \{ dx^j \}$$

V. Coordinate Transformation

23.8

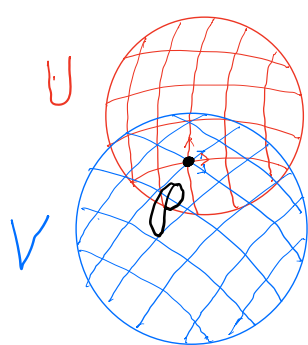
Let (φ_U, U) and (φ_V, V) be two coordinate charts surrounding P .

On UV one has two bases and their duals:

Both satisfy their duality relations at P :

$$\langle dx^i | \frac{\partial}{\partial x^i} \rangle = \delta^i_i \quad \text{and} \quad \langle dy^k | \frac{\partial}{\partial y^k} \rangle = \delta^k_k.$$

From page 22.8 one knows that the two vector bases are related by

$$\frac{\partial}{\partial x^i} = \frac{\partial(y^k)}{\partial x^i} \frac{\partial}{\partial y^k}.$$


$$\frac{\partial}{\partial x^i} = \frac{\partial(y^j)}{\partial x^i} \frac{\partial}{\partial y^j}$$

Figure 23.2: Relation between basis at P relative to U and that relative to V .

What does this relation imply about the relationship between the corresponding dual bases?

A: In $T_P^*(M)$ consider the covector

$$\frac{\partial(x^j)}{\partial y^l} dy^l.$$

Its value on $\frac{\partial}{\partial x^i}$ is

$$\begin{aligned} \langle \frac{\partial(x^j)}{\partial y^l} dy^l | \frac{\partial}{\partial x^i} \rangle &= \frac{\partial(x^j)}{\partial y^l} \langle dy^l | \frac{\partial}{\partial x^i} \rangle \\ &= \frac{\partial(x^j)}{\partial y^l} \frac{\partial(y^l)}{\partial x^i} \end{aligned}$$

Using the chain rule one obtains

$$\begin{aligned} &= \frac{\partial(x^j)}{\partial x^i} \\ &= \delta^j_i. \end{aligned}$$

Compare this equality with

$$\langle dx^j | \frac{\partial}{\partial x^i} \rangle = \delta^j_i$$

23.9

and conclude that

$$\frac{\partial \langle x^j \rangle}{\partial y^l} dy^l = dx^j$$

This is because

"Two linear functions are equal if and only if they have the same value when evaluated on a set of basis vectors."