LECTURE 24

I. Tangent to a Curve

II. Vector Field and Bundle Section

Optional Reading (See "Concluding Comment" on p 24.9) Chapter 4, pages 81-87; Chapter 8 in "Mathematical Methods of Classical Mechanics" by V. I. Arnold

I. Vector as tangent to curve

Referring to a vector
$$v = \alpha^i \frac{\partial}{\partial x^i}$$
merely in terms of (i) its components
$$v \rightsquigarrow (\alpha^1, \dots, \alpha^n) \in R^n$$
relative to a chosen (or given) coordinate basis,

or merely in terms of (ii) its properties as a derivation acting on scalar fuctions, is deficient: these referents do not tell what observed entities of reality are mathematized by a tangent vector.

This deficiency is removed by identifying a tangent vector as the directional derivative at a point along a curve in M passing through the isograms ("level surfaces") of a scalar function $f \in C^\infty(M, \varphi, R)$.

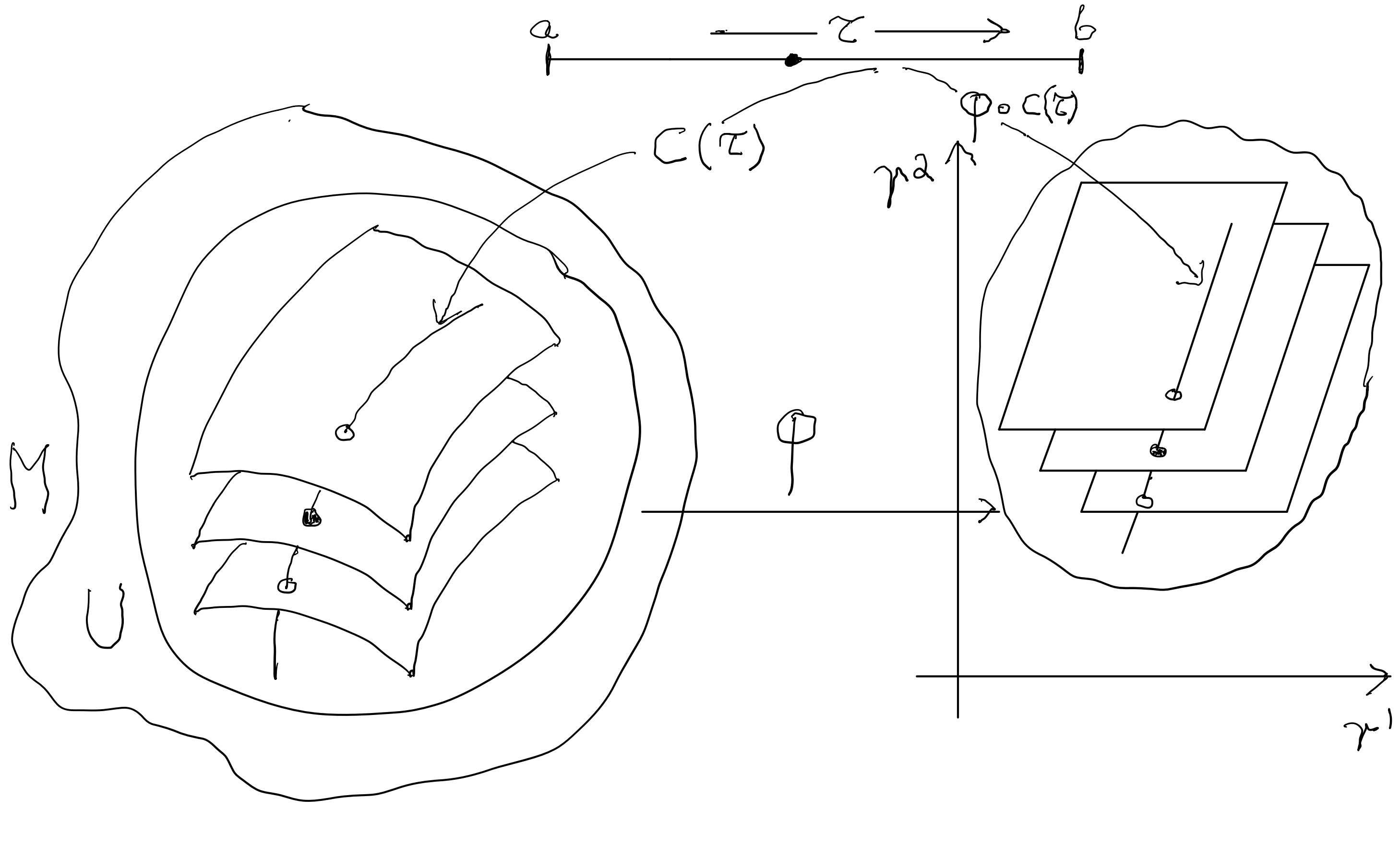

Figure 24.1: Curve $c(\tau)$ passing through the isograms of the function f.

Consider

1.) the curve $c(\tau)$ through point $P \in M$:

$$\begin{aligned} c:\ & R \longrightarrow M \\ & \tau \rightsquigarrow c(\tau) \end{aligned}$$

with its coordinate representative

$$\begin{aligned} \varphi\circ c:\ & R \longrightarrow R^n \\ & \tau \rightsquigarrow \varphi\circ c(\tau) \equiv \varphi(c(\tau)) \\ & \qquad = \left(\varphi^1(c(\tau)), \dots, \varphi^n(c(\tau))\right) \\ & \qquad = \left(c^1(\tau), \dots, c^n(\tau)\right) \\ & \qquad = \left(r^1(c^i(\tau)), \dots, r^n(c^i(\tau))\right) \end{aligned}$$

2.) a scalar function

$$f: \quad M \longrightarrow R$$

with its coordinate representative

$$\begin{aligned} f\circ\varphi^{-1}:\ & R^n \longrightarrow R \\ & (r^1, \dots, r^n) \rightsquigarrow f\circ\varphi^{-1}(r^1, \dots, r^n) \equiv f_{U\text{-rep}}(r^1, \dots, r^n) \end{aligned}$$

3.) the function f evaluated on the curve c, i.e. the composite $f\circ c$

$$\begin{aligned} f\circ c:\ & R \longrightarrow R \\ & \tau \rightsquigarrow f\circ c(\tau) = \underbrace{f\circ\varphi^{-1}} \circ \underbrace{\varphi\circ c(\tau)} \\ & \qquad = \overbrace{f_{U\text{-rep}}}\overbrace{\left(c^1(\tau), \dots, c^n(\tau)\right)} \end{aligned}$$

4.) the τ-derivative of this composite by using the chain rule

$$\frac{d}{d\tau} f\circ c(\tau) = \left.\frac{\partial f_{\text{o-rep}}(r^1,\dots,r^n)}{\partial r^i}\right|_{\varphi(c(\tau))} \cdot \frac{dc^i}{d\tau}$$

$$= \left.\frac{\partial(f\circ\varphi)}{\partial r^i}\right|_{\varphi(c(\tau))} \cdot \frac{dc^i}{d\tau}$$

$$\equiv \left.\frac{\partial(f)}{\partial x^i}\right|_{\varphi(c(\tau))} \cdot \frac{dc^i}{d\tau}$$

The tangent u to the curve $c(\tau)$ at the point $P_0 = c(\tau_0)$ is based on the four observations considered above. They are condensed into the following

Definition ("Tangent to a curve")

The tangent u to the curve $c(\tau)$ at $P_0 = c(\tau_0)$ is the map

$$u: \; C^\infty(M, P, R) \longrightarrow R$$

$$f \rightsquigarrow u(f) = \left.\frac{dc^i}{d\tau}\right|_{\tau_0} \left.\frac{\partial(f)}{\partial x^i}\right|_{P_0=c(\tau_0)}$$

This holds for <u>all</u> $f\in C^\infty(M,P,R)$. Thus

$$\boxed{\mathbb{u} = \frac{dc^i}{d\tau}\frac{\partial}{\partial x^i}}$$

$$\equiv \frac{d}{d\tau} = \left\{ \begin{array}{l} \text{"rate of change of an} \\ \text{as-yet-unspecified} \\ \text{function (= property)"} \end{array}\right.$$

One readily infers that u is a derivation, i.e. a tangent vector determined by $c(\tau)$.

II. Vector Field

Be it a force field, an electric field, or a fluid velocity field, their conceptual common denominator is that they are vector fields on a manifold. They can be defined on various levels of abstraction. We shall give two of them within the framework of modern (i.e post W.W.II) mathematics.

Definition ("Vector Field")
A vector field is an assignment a vector $\mathbb{v}$ to each point $P \in M$ of the manifold.

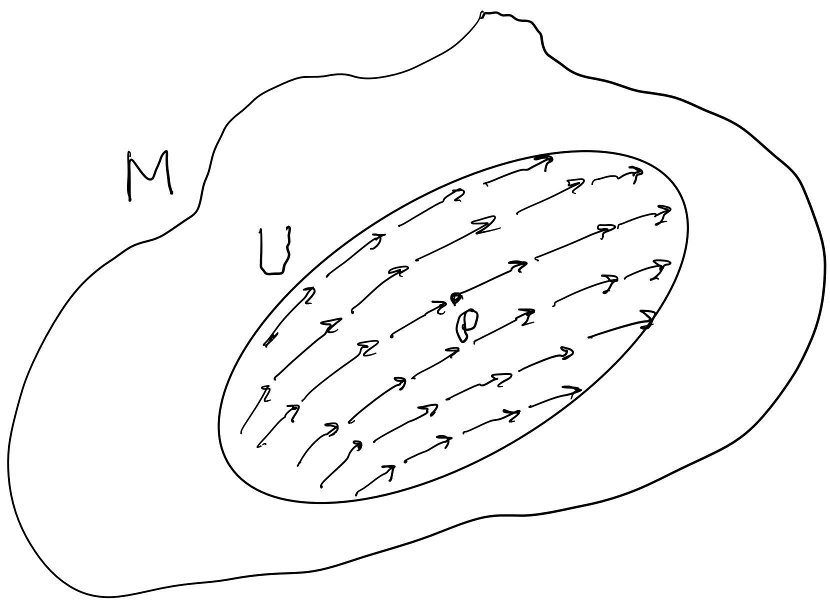

Figure 24.2: Vector field in the neighborhood U of the manifold M.

Relative to a given coordinate chart (φ_U, U), whose coordinate function are

$$x^i(P) = r^i \circ \varphi(P) \quad i=1,\cdots,n,$$

a vector field has the form

$$V = \alpha^i(x^1,\cdots,x^n)\frac{\partial}{\partial x^i}.$$

A tangent vector field V is said to be smooth whenever $V(f)$ is smooth for all $f \in C^\infty(M,P,R)$.

The second definition of a tangent vector field is based on viewing it as a graph in a tangent (vector) bundle of a manifold in the same way that a function is viewed as a graph in a Cartesian plane.

The graphing domain for a vector field tangent to a manifold is mathematized by a structure with the following

Definition ("Tangent bundle")

The tangent bundle $T(M)$ of a manifold M (Fig. 24.3) is the union of its disjoint tangent spaces $T_P(M)$ at all points $P \in M$,

$$T(M) = \bigcup_{P\in M} T_P(M)$$

together with its projection map

$$\begin{aligned} \pi:\ T(M) &\longrightarrow M \\ \left(P, v^i\frac{\partial}{\partial x^i}\Big|_P\right) &\rightsquigarrow \pi\left(P, v^i\frac{\partial}{\partial x^i}\right) = P \end{aligned}$$

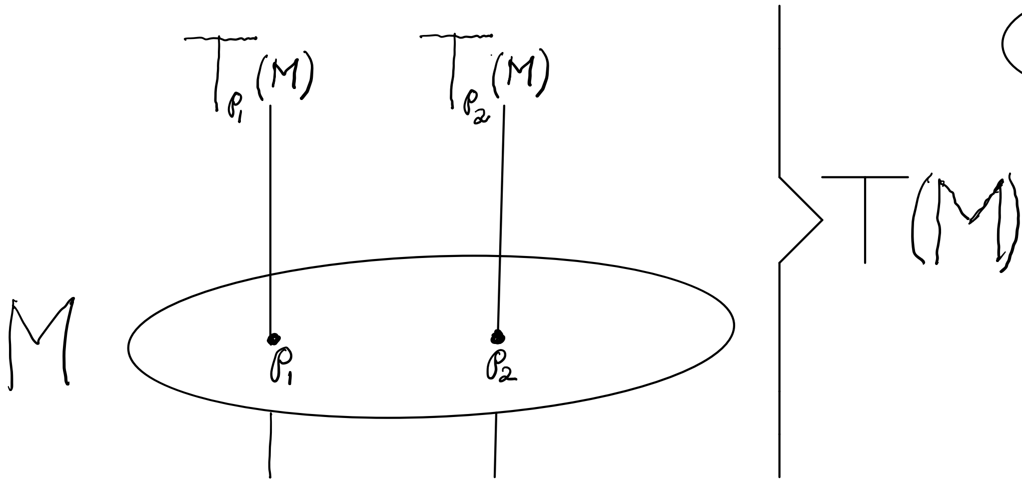

Figure 24.3: Tangent bundle $T(M)$ as the union of M's disjoint (nonoverlapping) tangent spaces.

Given the coordinate chart (φ_U, U), the coordinatization of $T(M)$ is achieved by means of

$$\Phi: \quad T(M) \longrightarrow U \times \mathbb{R}^n$$

$$(p, \mathbb{V}_p) = \left(p, v^i \frac{\partial}{\partial x^i}\Big|_p\right) \rightsquigarrow \Phi\left(p, v^i \frac{\partial}{\partial x^i}\Big|_p\right) = \left(p, (v^1, \dots, v^n)\right)$$

$$= \left((x^1, \dots, x^n), (v^1, \dots, v^n)\right) \in \mathbb{R}^n \times \mathbb{R}^n$$

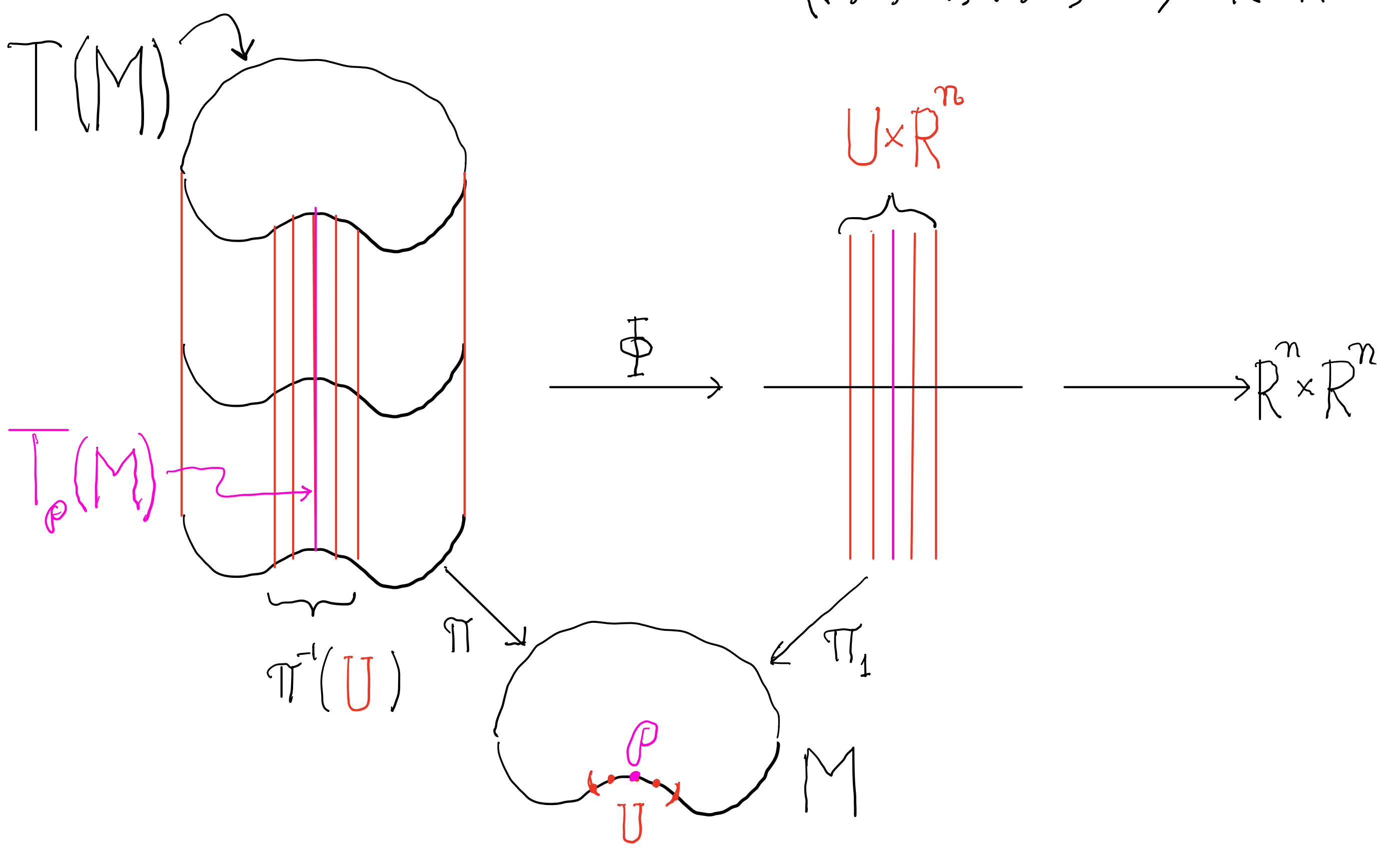

Figure 24.4: Tangent bundle $T(M)$ and its coordinate chart $(\Phi, \bigcup_{p \in U} T_p(M))$. The projection map $\pi = \pi_1 \circ \Phi$ on $\pi^{-1}(U)$ projects the "fiber" over U down onto U.

Comment

The vector field $v = v^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}$ on M is a graph in $T(M)$

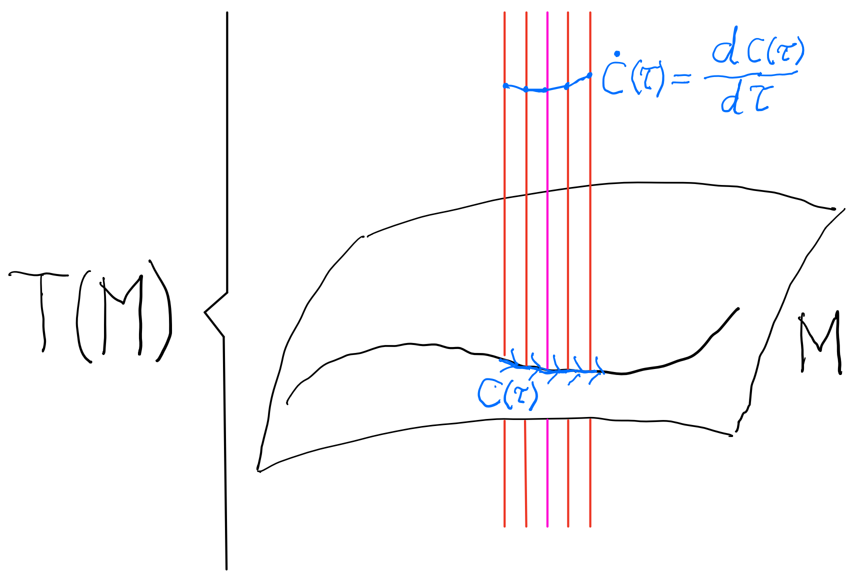

Figure 24.5: A curve $c(\tau)$ with its tangent vectors $\dot{c}(\tau)$ on M, when mathematized in $T(M)$, becomes graph $\dot{c}(\tau)$ situated in the "fiber" of the curve points $c(\tau)$ on M.

Concluding Comments.

1.) The tangent bundle is a geometrical and unit-economical way of mathematizing Lagrangian mechanics. The Lagrangian of a mechanical system is a scalar-valued function on the tangent bundle $T(M)$ of its configuration manifold M. V.I. Arnold in Chap. 4 of his "Mathematical

Methods of Classical Mechanics" highlights and also illustrates this point with a number of good examples. 24.9

2.) The cotangent bundle $T^*(M)$ of a base manifold is the union of its disjoint cotangent spaces $T_P^*(M)$, the vector spaces dual to $T_P(M)$, at each point P. Given a metric tensor at each P, one can transform the tangent bundle $T(M)$ into the cotangent bundle $T^*(M)$ of the mechanical system. This transforms the Lagrangian into the Hamiltonian mathematization of the mechanical system. The cotangent bundle formulation of Hamiltonian mechanics is the portal that leads from the local to the global analysis of mechanics in astrophysics, plasma physics and other studies of dynamical systems.

Again, V.I. Arnold in Chapter 8 of his "Mathematical Methods of Classical Mechanics" highlights also the cotangent bundle-based methods of Hamiltonian mechanics.