# LECTURE 26

I. Isograms of a function
vs
Curves of the flow field of a vector field

II. Taylor series on a curve of the flow field

III. The exponential map

IV. Vector as a displacement generator

In MTW read Box 8.4, Sect 9.2, 9.6

In Singer & Thorpe read p126 (or P142 in the Springer edition)

In mathematics the importance of a concept can be gauged by the number of contexts where it plays a central role. A vector $\mathbf{u}$ at $P$, because of its multi-faced nature is, such a concept. It is

(i) an element in $T_P(M)$ — Lecture 23

(ii) a derivation at $P$ — Lecture 24

(iii) the tangent to a curve at $P$ — Lecture 25

(iv) a vectorial displacement generator — Lecture 26

(v) the exponent in the exponential map $e^{\tau \mathbf{u}}$ — Lecture 26

# I. Scalar Field vs. Vector Field

Salar properties on a manifold are mathematized by scalar functions on a manifold. If $f(x^1, \dots, x^n)$ is a scalar function on $M$, then it physical referents are in the form of mathematical isograms.

By contrast, the flow field of a given vector field on $M$,

$$\mathbf{u}(x) = u^i(x^1, \dots, x^n)\frac{\partial}{\partial x^i},$$

is mathematized by means of the continuos family of orbits or trajectories, i.e. "integral curves," whose tangents are vectors of that preexisting vector field.

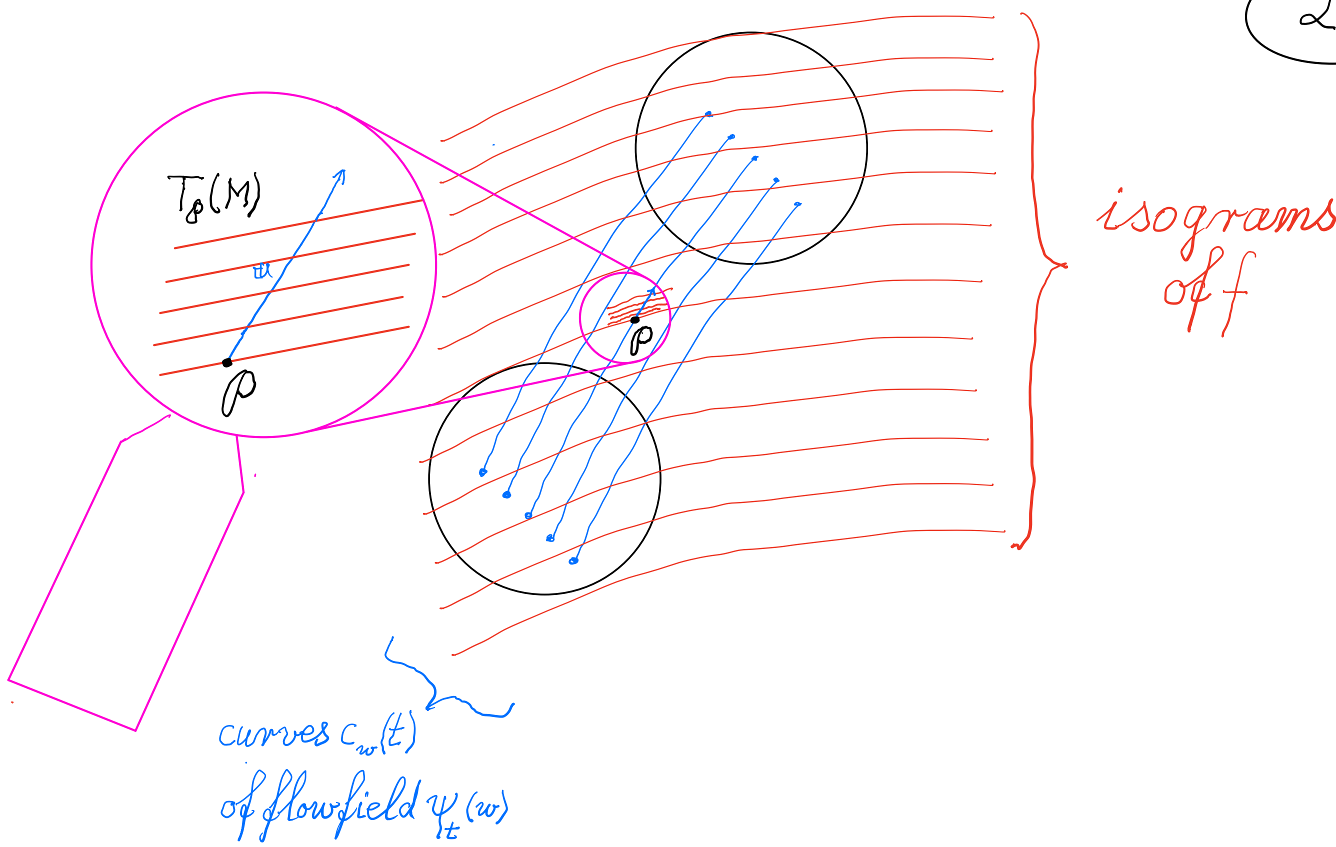


Figure 26.1: Integral curves $c_w(t)$ passing through the isograms of the scalar function $f(x)$.

The observed existence of scalar isograms and integral curves trace their origins back to the

Definition ("Scalar field")

A <u>smooth scalar field</u> $f \in C^\infty(M, R)$, namely the assignment to each point $P \in M$ a scalar $f(P) \in R$, is a family of $(n-1)$-dimensional <u>isograms</u> by a single variable, say $\gamma$:

$$f(x) = \gamma.$$

and to the

Definition ("Flow field")

The smooth flow field of a smooth vector field $\mathbf{u} \in C^{\infty}(M, T(M))$, namely the assignment to each point $P \in M$ a vector $\mathbf{u}(P) \in T_P(M)$, is a family of integral curves parametrized by $w \in \mathbb{R}^n$:

$$c_w(t) = \{c^i(t; w^1, \dots, w^n\} = \{c^i_w(t)\}$$

II. Taylor series of a scalar function on a curve.

In non-linear mathematic a scalar field and a vector field are paired concepts corresponding to the duality between linear functions ("covectors") and vectors in linear mathematics.
This scalar-vector pairing is achieved by the method of the Taylor series as follows:

1.) Consider the solution curve, say $c(t)$, to the differential equation

$$\frac{d}{dt} = \mathbf{u}$$

i.e. to

$$\frac{dc^i(t)}{dt}\frac{\partial}{\partial x^i} = u^i(c^1(t), \dots, c^n(t)\}\frac{\partial}{\partial x^i}$$

with starting point

$$c_w(t=0) = w \equiv (w^1, \dots, w^n)$$

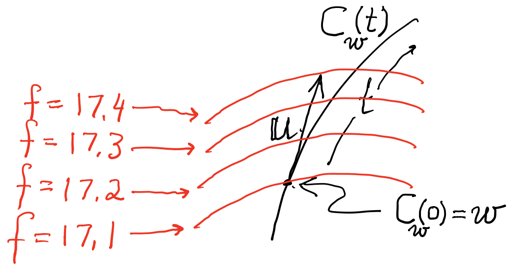


Figure 26.2: Integral curve $c(t)$ passing through the isograms of $f$

2.) Consider a scalar field $f(x)$. Evaluate it on the solution curve $c_w(t)$ and obtain the single variable function

$$f(c_w(t)) = h(t)$$

3.) Consider its Taylor series expansion around $t=0$ at $c_w(0)=w$:

$$f(c_w(t)) = h(0) + t\left.\frac{dh}{dt}\right|_{t=0} + \frac{t^2}{2!}\frac{d^2h(0)}{dt^2} + \cdots$$

$$f(c^j_w(t)) = f(c^j_w(0)) + t\left.\frac{dc^i_w}{dt}\right|_{t=0}\left.\frac{\partial f}{\partial x^i}\right|_{c^i_w(0)} + \frac{t^2}{2}\left.\frac{d}{dt}\frac{d}{dt}h(t)\right|_{t=0}$$

In terms of the directional derivative

$$\frac{d}{dt} = \frac{dc^i}{dt}\frac{\partial}{\partial x^i} = u^i\frac{\partial}{\partial x^i} = D_u = u$$

one has

$$f(c^j_w(t)) = f(w) + t\,D_u f(w) + \frac{t^2}{2!}D_u D_u f(w) + \cdots$$

$$= f(w) + t\,u f(w) + \frac{t^2}{2!}u u f(w) + \cdots$$

$$= \left(1 + t u + \frac{t^2}{2}u u + \cdots\right) f(x)\Big|_{x=w}$$

Thus, the value $f$ along any point along the curve $c_w(t)$ is

$$\boxed{f(c_w(t)) = \exp(t u) f(w)} \qquad (26.1)$$

# Comment

The justification for introducing the exponential operator $e^{t\mathcal{U}}$ comes from its additive property

$$\boxed{\exp\big((\tau+\tau')\mathcal{U}\big) = \exp(\tau\mathcal{U})\,\exp(\tau'\mathcal{U})} \qquad (26.2)$$

From Eqs. (25.2) and (25.12) [in Lecture 25] we have

$$c_w(t) = \psi_t(w)$$

$$\psi_{\tau+\tau'}(w) = \psi_\tau \circ \psi_{\tau'}(w)$$

In order to verify Eq. (26.2), use Eq. (26.1) three times in conjunction with these two equations from Lecture 25.

$$\exp\big((\tau+\tau')\mathcal{U}\big) f(w) \underset{①}{=} f\big(c_w(\tau+\tau')\big) = f\big(\psi_{\tau+\tau'}(w)\big)$$

$$= f\circ\psi_{\tau+\tau'}(w)$$

$$= f\circ\psi_\tau\circ\psi_{\tau'}(w)$$

$$= f\circ\psi_\tau\big(\psi_{\tau'}(w)\big)$$

$$= f\circ\psi_\tau\big(c_w(\tau')\big)$$

$$\overset{②}{=} \exp(\tau'\mathcal{U})\, f\circ\psi_\tau(w)$$

$$= \exp(\tau'\mathcal{U})\, f\big(\psi_\tau(w)\big)$$

$$= \exp(\tau'\mathcal{U})\, f\big(c_w(\tau)\big)$$

$$\overset{③}{=} \exp(\tau'\mathcal{U})\,\exp(\tau\mathcal{U})\, f(w)$$

This holds for all smooth functions at the point $w\in M$, i.e. $\forall f\in C^\infty(M,w,\mathbb{R})$.
Thus Eq. (26.2) is true indeed.

# IV. Vector as a displacement generator.

Consider two points

$$C_w(0) = w = \mathcal{P}$$

$$C_w(\tau) = \bar{\mathcal{P}} \equiv \mathcal{P} + \Delta\mathcal{P} \qquad \text{("displaced point")}$$

on the curve $c_w(t)$ whose tangent at $w = \mathcal{P}$ is

$$u = u^i(x^1, \dots, x^n)\, \frac{\partial}{\partial x^i}\bigg|_{\{x^j\} = \{w^j\}}$$

and the function

$$f(x^1, \dots, x^n) = x^k$$

with its two isograms $f = a^k$ and $f = \bar{a}^k = a^k + \Delta x^k$ running through these two points as depicted in Figure 26.3.

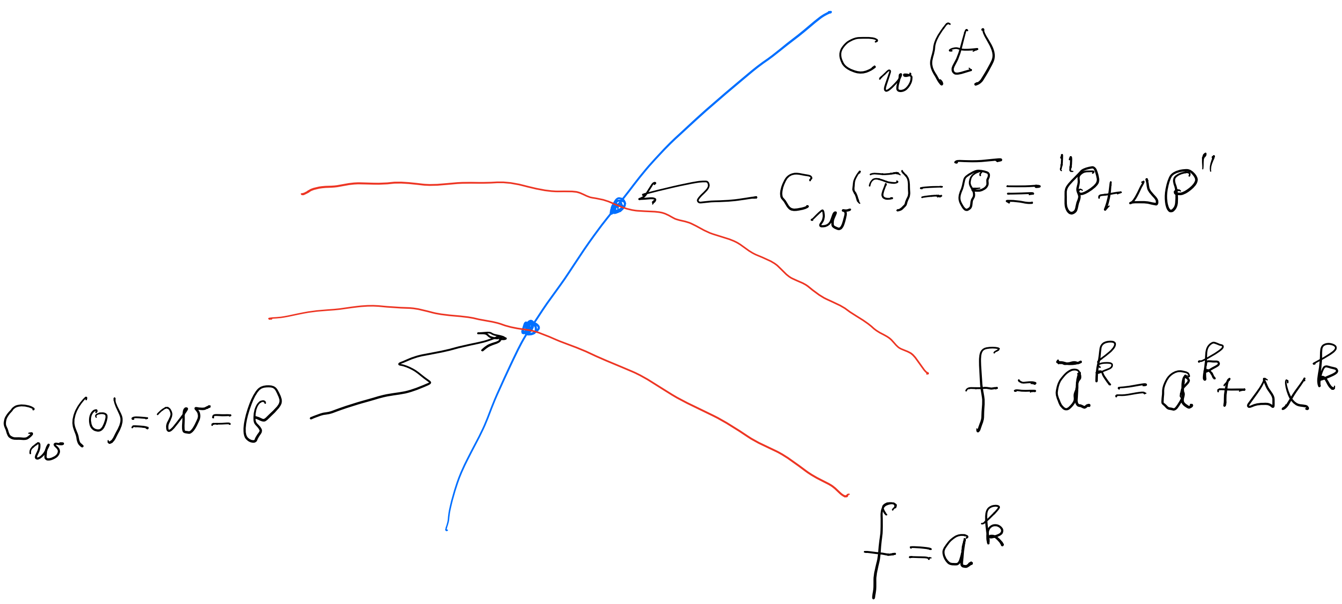


Figure 26.3: Two points, $\mathcal{P}$ and the displaced point $\bar{\mathcal{P}} = \mathcal{P} + \Delta\mathcal{P}$, on a given curve $c_w(t)$, with two isograms of the coordinate function $f(x^1, \dots, x^n) = x^k$ running through them.

The value of $f(\bar{\mathcal{P}})$ is related to that of $f(\mathcal{P})$ by means of the Taylor series expansion

$$f(\bar{\mathcal{P}}) = \exp(\bar{\tau}\,\mathbf{u})\,f(\mathcal{P})$$

$$f(\mathcal{P}+\Delta\mathcal{P}) = f(\mathcal{P}) + \bar{\tau}\,\mathbf{u}\,f(\mathcal{P}) + \frac{(\bar{\tau})^2}{2!}\,\mathbf{u}\mathbf{u}\,f(\mathcal{P}) + \cdots$$

$$= f(w^j) + \bar{\tau}\,u^i \frac{\partial f}{\partial x^i}\bigg|_{\{w^\ell\}} + \frac{(\bar{\tau})^2}{2!}\,u^i \frac{\partial}{\partial x^i}\left(u^j \frac{\partial f}{\partial x^j}\right)\bigg|_{\{w^\ell\}} + \cdots$$

Letting $f = x^k$, one obtains

$$x^k + \Delta x^k = x^k + \bar{\tau}\,u^i \frac{\partial x^k}{\partial x^i} + \frac{\bar{\tau}^2}{2!}\,u^i \frac{\partial}{\partial x^i}\left(u^j \frac{\partial x^k}{\partial x^j}\right) + \cdots$$

or

$$\Delta x^k = \bar{\tau}\,u^k + \frac{(\bar{\tau})^2}{2!}\,u^i \frac{\partial (u^k)}{\partial x^i} + \cdots$$

Thus $\{\Delta x^k\}$ are the coordinate differences of the displacement from $\mathcal{P}$ to $\mathcal{P}+\Delta\mathcal{P}$. If they lie on the integral curve whose tangent at $\mathcal{P}$ is $\mathbf{u}$, then

$$\Delta x^k = \Delta\mathcal{P}(x^k) = \bar{\tau}\,\mathbf{u}(x^k) + \frac{(\bar{\tau})^2}{2!}\,\mathbf{u}\mathbf{u}(x^k) + \cdots,$$

or leaving the $x^k$'s as-yet-unspecified,

$$\Delta\mathcal{P} = \bar{\tau}\,\mathbf{u}\big|_{\mathcal{P}} + \underbrace{\frac{(\bar{\tau})^2}{2!}\,\mathbf{u}\,\mathbf{u}\big|_{\mathcal{P}}}_{\text{neglegible for } |\bar{\tau}| \ll 1} + \cdots$$

The first term of the displacement $\Delta\mathcal{P}$ is the principal linear part. It is a vector. For $|\bar{\tau}| \ll 1$ the 2nd non-linear term is not a vector. This is because it is not a derivation:

$$\mathbf{u}\mathbf{u}(fg) \neq f\,\mathbf{u}\mathbf{u}(g) + \mathbf{u}\mathbf{u}(f)\,g.$$