LECTURE 27

I. Commutator of two vector fields

II. Cotangent space of differentials

III. Coordinate-induced dual basis

IV. Physical meaning of a differential and of a tangent vector

In MTW read Box 9.2, Sections 9.6, 9.3, 9.4

Memorize Eq. (9.14).

I. Commutator $[u,v]$ of two vector fields

Consider two vector fields u and v and their intersecting integral curves $c_{u(P)}(\tau)$ and $c_{v(P)}(s)$ in a neighborhood of P.

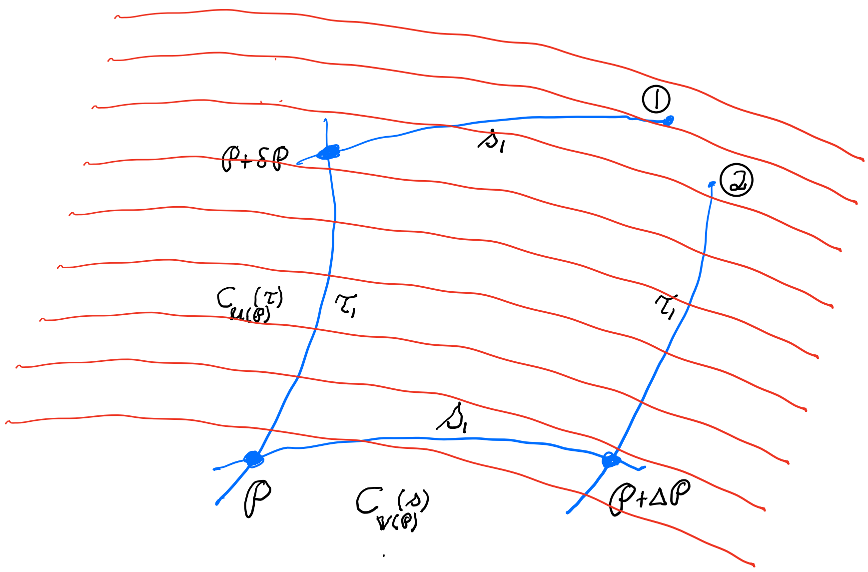

Figure 27.1: Integral curves $c_u(\tau)$ and $c_v(s)$ of vector fields u and v passing through the isograms of scalar function f.

Consider two of these curves which, as depicted in Figure 27.1, intersect at point P. Together with their respective opposing curve segments they form the quadrilateral $(P, P+\delta P, 1, 2, P+\Delta P)$. This quadrilateral is in general not closed. Its non-zero gap in Figure 27.1 is formed by the end points ① and ② of the two curve segments.

The environment of this open quadrilateral is permeated by a scalar function $f \in C^\infty(M, P, \mathbb{R})$ with typical isograms as depicted in Figure 27.1.

The system of two vector fields u and v together with the set of ambient smooth scalar fields, such as f, gives rise to the concept of the commutator $[u,v]$ of u and v by asking and answering the following three questions:

1. $f(①) = ?$
2. $f(②) = ?$
3. $f(①) - f(②) = ?$

They are answered using the Taylor series expansion along

$$\text{and} \qquad \begin{array}{l} ① \longrightarrow P + \delta P \longrightarrow P \\ ② \longrightarrow P + \Delta P \longrightarrow P \end{array}$$

To 2^{nd} order accuracy one has along $① \longrightarrow P + \delta P \longrightarrow P$

$$f(①) = f\Big|_{P+\delta P} + s_1 v f\Big|_{P+\delta P} + \frac{s_1^2}{2!} v v f\Big|_{P+\delta P} + \cdots$$

$$= f\Big|_P + \tau_1 u f\Big|_P + \frac{\tau_1^2}{2} u u f\Big|_P + \cdots$$

$$s_1 v f\Big|_P + s_1 \tau_1 u v f\Big|_P + \cdots + \frac{s_1^2}{2} v v f\Big|_P + \cdots$$

Similarly, to 2^{nd} order one has along $② \longrightarrow P + \Delta P \longrightarrow P$

$$f(②) = f\Big|_{P+\Delta P} + \tau_1 u f\Big|_{P+\Delta P} + \frac{\tau_1^2}{2!} u u f\Big|_{P+\Delta P} + \cdots$$

$$= f\Big|_P + s_1 v f\Big|_P + \frac{s_1^2}{2} v v f\Big|_P + \cdots$$

$$\tau_1 u f\Big|_P + \tau_1 s_1 v u f\Big|_P + \cdots + \frac{\tau_1^2}{2} u u f\Big|_P + \cdots$$

$f(①) - f(②)$ defines a vector; the commutator $[u,v]$ at P:

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$$\frac{f(①)-f(②)}{s, \tau_1} = (uv - vu)f\Big|_P \equiv [u,v]f\Big|_P$$

Comment

(i) $[u,v]$ is a derivation and hence a vector:
$[u,v](f\,g) = f[u,v]g + [u,v](f)\,g$, which is the (to-be-verified) product ("Leibnitz") rule.

(ii) If $u = u^\alpha \frac{\partial}{\partial x^\alpha}$ and $v = v^\beta \frac{\partial}{\partial x^\beta}$, then $[u,v] = [u,v]^\gamma \frac{\partial}{\partial x^\gamma}$, where the component $[u,v]^\gamma$ depend on the (to-be-computed) partial derivatives of u^α and v^β!

II. The Cotangent space $T_P^*(M)$

A) Vector Space

Every vector space V has its space of duals, the covector space V^*. Furthermore, given a basis $B = \{e_1, \cdots, e_n\}$ for V, its dual basis $B^* = \{\underset{\sim}{\omega}^1, \cdots, \underset{\sim}{\omega}^n\}$ is determined by the duality relation,

$$\langle \underset{\sim}{\omega}^j | e_i \rangle = \delta^j{}_i \, .$$

Every point P of a manifold has its tangent vector space

$$V_P = T_P(M) \, .$$

Relative to a coordinate system $x^\delta(P) = r^\delta \circ \varphi(P)$ the induced basis for $T_P(M)$ is

$$B_p = \left\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right\}$$

so that

$$T_p(M) = \text{span}\left(\left\{\frac{\partial}{\partial x^i}\right\}\right)$$

B) Space of Duals to $T_p(M)$

The space $T_p^*(M)$ of duals to $T_p(M)$ is the space of differentials at at p. A differential arises from observations which are condensed into the following.

Definition ("The differential of a scalar function")

Consider $u, v \in T_p(M)$

$f, g \in C^\infty(M, p, R)$

$\alpha, \beta \in R$;

then the differential df of f is the following linear function

$$df : T_p(M) \longrightarrow R$$
$$u \rightsquigarrow \langle df | u \rangle = u(f)$$

Note1: df is <u>linear</u> on $T_p(M)$. Indeed,

$$\begin{aligned} \langle df \,|\, \alpha u + \beta v \rangle &= (\alpha u + \beta v)(f) \\ &= \alpha u(f) + \beta v(f) \\ &= \alpha \langle df | u \rangle + \beta \langle df | v \rangle \end{aligned}$$

$\therefore$ $\boxed{df \in T_p^*(M) \text{, the set of linear functions on } T_p(M)}$

Note 2: The operator d [which maps functions to linear functions on $T_p(M)$] is linear, i.e. the differential of a linear combination of functions equals a linear combination of their differentials. This observation is mathematizes by the following

Theorem ("Linearity of d")

$$d(\alpha f + \beta g) = \alpha\, df + \beta\, dg$$

proof:
$$\begin{aligned}\langle d(\alpha f + \beta g) | u \rangle &= u(\alpha f + \beta g) \\ &= \alpha\, u(f) + \beta\, u(g) \\ &= \alpha \langle df | u \rangle + \beta \langle dg | u \rangle \quad \forall u \in T_p(M) \\ &= \langle \alpha\, df + \beta\, dg | u \rangle \quad \because \text{lin. functions are closed under lin. combinations}\end{aligned}$$

This holds $\forall u \in T_p(M)$. Thus,

$$\alpha\, df + \beta\, dg = d(\alpha f + \beta g)$$

Consequently, a linear combination of differentials is another differential in $T_p^*(M)$. Thus $T_p^*(M)$, the space of differentials, is a vector space, <u>the cotangent space</u>.

III. Dual basis for $T_p^*(M)$

Let $f^1(x^1, \ldots, x^n) = x^1$
$$\vdots$$
$f^n(x^1, \ldots, x^n) = x^n$

Conclusion: $\{dx^1, \ldots, dx^n\}$ is a <u>basis</u> for $T_p^*(M)$.

a) $\{dx^j\}$ has the <u>spanning property</u>

Indeed, let $df \in T_p^*(M)$. Then

$$\begin{aligned}\langle df | u\rangle &= u(f) = u^i \frac{\partial f}{\partial x^i} \\ &= u^i \delta_i^j \frac{\partial f}{\partial x^j} \\ &= u^i \frac{\partial x^j}{\partial x^i} \frac{\partial f}{\partial x^j} \\ &= \frac{\partial f}{\partial x^j} \langle dx^j | \frac{\partial}{\partial x^i} \rangle u^i \\ &= \langle \frac{\partial f}{\partial x^j} dx^j | u \rangle \quad \forall u \in T_p(M)\end{aligned}$$

Thus, $df = \frac{\partial f}{\partial x^j} dx^j$, i.e. $\{dx^j\}$ is a spanning set

b) The set $\{dx^j\}$ is <u>linearly independent</u>

Consider

$$\alpha_j \, \underset{\sim}{dx^j} = \underset{\sim}{0}.$$

Evaluate it on $e_i = \frac{\partial}{\partial x^i}$ and obtain

$$\begin{aligned}0 = \langle \underset{\sim}{0} | e_i \rangle &= \langle \alpha_j dx^j | \frac{\partial}{\partial x^i} \rangle \\ &= \alpha_j \delta_i^j \\ &= \alpha_i \qquad i = 1, \dots, n\end{aligned}$$

Conclusion

$B^* = \{dx^1, \dots, dx^n\}$ is the coordinate induced basis for $T_p^*(M)$ which is dual to the basis

$B = \{e_1 = \frac{\partial}{\partial x^1}, \dots, e_n = \frac{\partial}{\partial x^n}\}$ for $T_p(M)$ and

$$\langle dx^j | e_i \rangle = \langle dx^j | \frac{\partial}{\partial x^i} \rangle = \delta_i^j$$

which is the duality relationship.

IV. Physical meaning of df and $\mathbf{u}$.

Evaluate f on the curve $c(t)=\{c^i(t)\}$ whose tangent at $P=c(t)$ is $\dot{c}(t)=\frac{dc^i}{dt}\frac{\partial}{\partial x^i}\Big|_P = u^i\frac{\partial}{\partial x^i}\Big|_P = \mathbf{u}\Big|_P$; and find that

$$\frac{d(f)}{dt} = \frac{\partial(f)}{\partial x^i}u^i = \underbrace{\langle df|\mathbf{u}\rangle}_{①} = \underbrace{\mathbf{u}(f)}_{②}.$$

① Omitting explicit reference to $\mathbf{u}$, one has

df = "rate of change of f into an as-yet-unspecified direction."

② Omitting explicit reference to f, one has

$\mathbf{u}=\frac{d}{dt}$ = "instantaneous rate of change (at a point on a curve) of an as-yet-unspecified scalar function.