LECTURE 28

I. Maxwell's Field Equations

II. Exterior Algebra

III. Exterior Calculus

In MTW read Chapter 3, especially Sections 3.4 and 3.5

I. Mathematications of the Maxwell Field Equations

a) The integral form of the Maxwell field equations is

$$\oiint \vec{E}\cdot d^2\vec{S} = \iiint 4\pi\rho\, d^3x \qquad \oiint \vec{B}\cdot d^2\vec{S} = 0$$

$$\oint \vec{H}\cdot d\vec{l} = \iint \left(4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t}\right)\cdot d^2\vec{S} \qquad \oint \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\iint \vec{B}\cdot d^2\vec{S}$$

Maxwell

b) Their differential formulation is

$$\nabla\cdot\vec{E} = 4\pi\rho \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = 4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Maxwell

c) Their spacetime tensor component formulation is

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \qquad F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

} Minkowski

d) Their post-WWII formulation in terms of exterior mathematics is

$$d * \underset{\sim}{F} = 4\pi * \underset{\sim}{J} \qquad d\underset{\sim}{F} = 0$$

i.e.

$$d(*F_{|\alpha\beta|}\, dx^\alpha \wedge dx^\beta) = 4\pi * J_{|\alpha\beta\gamma|}\, dx^\alpha\wedge dx^\beta\wedge dx^\gamma \qquad d(F_{\alpha\beta}\, dx^\alpha\wedge dx^\beta) = 0$$

Cartan

Exterior mathematics consists of exterior algebra and exterior calculus

II. Exterior Algebra

The mathematization of magnetic flux, particle flux in 3-d or 4-d spacetime, the electromagnetic flux in 4-d, and others is in terms of antisymmetric tensors, linear combinations of totally antisymmetric tensor products. They are "wedge products" formed from basis elements

① Wedge Product as a Linear Combination of Tensor Products.

Following MTW's notation in Sections 3.5 and 9.5, let $\{\omega^\mu = dx^\mu\}_{\mu=0}^{3}$ be a coordinate induced basis for $T_p^*(M)$.

Let $\alpha = \alpha_\mu \omega^\mu$, $\beta, \gamma \in T_p^*(M)$. Then we have the following

Definition ("Wedge product")

(i) $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha = \frac{1}{2}(\alpha_\mu \beta_\nu - \beta_\mu \alpha_\nu)\, \omega^\mu \otimes \omega^\nu$

(ii) $\alpha \wedge \beta \wedge \gamma = \alpha \otimes \beta \otimes \gamma + \beta \otimes \gamma \otimes \alpha + \gamma \otimes \alpha \otimes \beta - \gamma \otimes \beta \otimes \alpha - \beta \otimes \alpha \otimes \gamma - \alpha \otimes \gamma \otimes \beta$

are the <u>wedge products</u> of these covectors. The symbol "$\wedge$" is called the "wedge" or the "exterior product" sign, or the "antisymmetric product" sign.

② Wedge Product as an "Exterior Product".

Let $\omega^1, \dots, \omega^n$ be a basis for V^*. Then

$$\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$$

is a basis for Λ^p, the tensor space of <u>totally antisymmetric tensors of rank $\binom{0}{p}$</u>, a.k.a. <u>p-forms</u>. The coordinate components of $\underset{\sim}{\alpha} \in \Lambda^p$ are (c.f. Lecture 15)

$$\underset{\sim}{\alpha}(e_{j_1}, \dots, e_{j_p}) \equiv \alpha_{j_1 \cdots j_p} .$$

They also have the property of being totally antisymmetric, namely

$$\alpha_{\pi(i_1 \cdots i_p)} = (-1)^\pi \alpha_{i_1 \cdots i_p} ,$$

where π is a permutation of p symbols, and $(-1)^\pi$ is $+1$ if π is an even permutation or -1 if π is an odd permutation. This second condition is the same as $\alpha_{i_1\cdots i_p}$ changing sign when any two sybscripts are interchanged.

It follows that

$$\underset{\sim}{\alpha} = \frac{1}{p!}\,\alpha_{i_1\cdots i_p}\;\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$
$$= \alpha_{|i_1\cdots i_p|}\;\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$$

Here vertical bars refer to the <u>restricted sum</u> with $i_1 < i_2 < \cdots < i_p$ only.

This constellation of ideas is illustrated by an antisymmetric tensor of rank $\binom{0}{2}$ in three dimensions.

$$\frac{1}{2!}\,\alpha_{i_1 i_2}\,\omega^{i_1}\wedge\omega^{i_2} = \frac{1}{2!}\left[\alpha_{12}\,\omega^1\wedge\omega^2 + \alpha_{21}\,\omega^2\wedge\omega^1 + \alpha_{23}\,\omega^2\wedge\omega^3 + \alpha_{32}\,\omega^3\wedge\omega^2 + \alpha_{31}\,\omega^3\wedge\omega^1 + \alpha_{13}\,\omega^1\wedge\omega^3\right]$$
$$= \frac{1}{2!}\left[(\alpha_{12}-\alpha_{21})\,\omega^1\wedge\omega^2 + (\alpha_{23}-\alpha_{32})\,\omega^2\wedge\omega^3 + (\alpha_{31}-\alpha_{13})\,\omega^3\wedge\omega^1\right]$$
$$= \alpha_{12}\,\omega^1\wedge\omega^2 + \alpha_{23}\,\omega^2\wedge\omega^3 + \alpha_{13}\,\omega^1\wedge\omega^3$$
$$= \alpha_{|i_1 i_2|}\;\omega^{i_1}\wedge\omega^{i_2}$$

The rank of an antisymmetric tensor can be increased by <u>exterior multiplication</u>. The result of such a multiplication is a <u>wedge product</u>. This multiplication (unlike, for example, the familiar cross product of vectors) is associative and is defined by its three distinguishing properties:

(i) $(a\underset{\sim}{\alpha} + b\underset{\sim}{\beta})\wedge\underset{\sim}{\gamma} = a\,\underset{\sim}{\alpha}\wedge\underset{\sim}{\gamma} + b\,\beta\wedge\underset{\sim}{\gamma}$ where $\underset{\sim}{\alpha},\underset{\sim}{\beta}\in\Lambda^p;\ \underset{\sim}{\gamma}\in\Lambda^q$

(ii) $(\underset{\sim}{\alpha}\wedge\underset{\sim}{\beta})\wedge\underset{\sim}{\gamma} = \underset{\sim}{\alpha}\wedge(\underset{\sim}{\beta}\wedge\underset{\sim}{\gamma}) \equiv \underset{\sim}{\alpha}\wedge\underset{\sim}{\beta}\wedge\underset{\sim}{\gamma}$ where $\underset{\sim}{\alpha}\in\Lambda^p,\ \underset{\sim}{\beta}\in\Lambda^q,\ \underset{\sim}{\gamma}\in\Lambda^r$

(iii) but $\underset{\sim}{\alpha}\wedge\underset{\sim}{\beta} = (-1)^{pq}\,\underset{\sim}{\beta}\wedge\underset{\sim}{\alpha}$

where $\underset{\sim}{\alpha}$ is a p-form

and $\underset{\sim}{\beta}$ is a q-form

III. Exterior Calculus

The exterior calculus is based on the exterior derivative operator d.
Let $F^p(U)$ be the collection of p-forms on a neighborhood $U \subset M$.
The observations about the calculus of such antisymmetric tensors fields are condensed into the following

Definition ("Exterior derivative")

The exterior differential operator d is the linear map

$$d: \quad F^p(U) \longrightarrow F^{p+1}(U)$$

$$p\text{-form} \rightsquigarrow (p+1)\text{-form}$$

with the following Four ("mechanical") Rules for Extrior Differentiation

(i) $d(\omega+\eta) = d\omega + d\eta$

(ii) $d(\omega^p \wedge \omega^q) = d\omega^p \wedge \omega^q + (-1)^p \omega^p \wedge d\omega^q$

(iii) for any ω, $dd\omega = 0$

(iv) for any function f, $df = \frac{\partial f}{\partial x^i} dx^i$

Comment.

A function is a tensor field of rank zero; hence it is a zero form $(p=0)$.

Exterior differentiation using the four rules is illustrated by the following

Example 1

Consider the 1-form

$$\omega = P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Then

$$d\omega = \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz\right)\wedge dx$$

$$+\left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz\right)\wedge dy$$

$$+\left(\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz\right)\wedge dz$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$$

Example 2

Consider the 2-form

$$\sigma = A\, dy\wedge dz + B\, dz\wedge dx + C\, dx\wedge dy .$$

Then

$$d\sigma = \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz\right)\wedge dy\wedge dz$$

$$+\left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy + \frac{\partial B}{\partial z}dz\right)\wedge dz\wedge dx$$

$$+\left(\frac{\partial C}{\partial x}dx + \frac{\partial C}{\partial y}dy + \frac{\partial C}{\partial z}dz\right)\wedge dx\wedge dy$$

$$= \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) dx\wedge dy\wedge dz$$

Example 3

Consider the p-form

$$\underset{\sim}{\sigma}^p = \sigma\, \underbrace{dx^{h_1}\wedge\cdots\wedge dx^{h_p}} \equiv \sigma\, \underbrace{dx^H}$$

("unit-economy")

and the q-form

$$\underset{\sim}{\omega}^q = \omega\, dx^{j_1}\wedge\cdots\wedge dx^{j_p}.$$

Then

$$d(\underset{\sim}{\sigma}^p\wedge\underset{\sim}{\omega}^q) = d(\sigma\omega\, dx^H\wedge dx^J)$$

$$= d(\sigma\omega) \wedge dx^H \wedge dx^J$$

$$= \frac{\partial(\sigma\omega)}{\partial x^i} dx^i \wedge dx^H \wedge dx^J$$

$$= \left(\frac{\partial\sigma}{\partial x^i}\omega + \sigma\frac{\partial\omega}{\partial x^i}\right) dx^i \wedge dx^H \wedge dx^J$$

$$= \frac{\partial\sigma}{\partial x^i} dx^i \wedge dx^H \wedge \omega\, dx^J + \sigma\, dx^i \wedge dx^H \wedge \frac{\partial\omega}{\partial x^i} \wedge dx^J$$

$$\boxed{d(\underset{\sim}{\sigma}^p \wedge \underset{\sim}{\omega}^q) = d\underset{\sim}{\sigma}^p \wedge \underset{\sim}{\omega}^q + (-1)^p \underset{\sim}{\sigma}^p \wedge d\omega^q}$$

Example 4 ("Poincare's Lemma")

Consider the p-form

$$\underset{\sim}{\sigma}^p = \sigma\, dx^{h_1} \wedge \cdots \wedge dx^{h_p} \equiv \sigma\, dx^H.$$

Then

$$d\underset{\sim}{\sigma}^p = \frac{\partial\sigma}{\partial x^i} dx^i \wedge dx^H$$

$$d(d\underset{\sim}{\sigma}^p) = \frac{\partial^2\sigma}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^H$$

$$= 0$$ from MTW's Problem 3.11 (Homework IV)