LECTURE 3

Geometry of Euclidean space
vs.
Physics of Lorentzian spacetime

1. Commensurable coordinates
2. Euclidean distance vs. Lorentzian interval
3. Euclidean vs Lorentzian data reduction
4. Euclid vs Lorentz: Perpendicular vs. simultaneous
5. Euclidean vs Lorentzian rotations
   (i) Why linear?
   (ii) Sine and cosine vs slope coefficients

READING ASSIGNMENT

(1) [T-W $\underbrace{5.2, 5.3}_{\text{2nd edition}}$ $\underbrace{(1.1, 1.6)}_{\text{1st edition}}$]

(2) Euclidean vs. Lorentz transformations
[T-W $\underbrace{L.3, L.4, L.5, L.6}_{\text{2nd edition}}$ $\underbrace{(1.8, 1.9)}_{\text{1st edition}}$

3.1b

1. Distance in Euclidean space vs. interval in Lorentzian spacetime via commensurable coordinates
2. Euclidean data reduction yields circles
   Lorentzian data reduction yields hyperbolas
3. Euclidean perpendicularity vs
   Lorentzian simultaneity
4. Euclidean vs. Lorentz rotations
   Their linearity
   Their coefficients.

Invariance of Euclidean Distance under reorientation of coordinate axes

I. Commensurate vs. Incommensurate Coordinate Lines.

The concept "distance" is a special case of the separation between two points. Consider a 2-dimensional domain coordinatized by kilometers in one direction and english miles in the other.

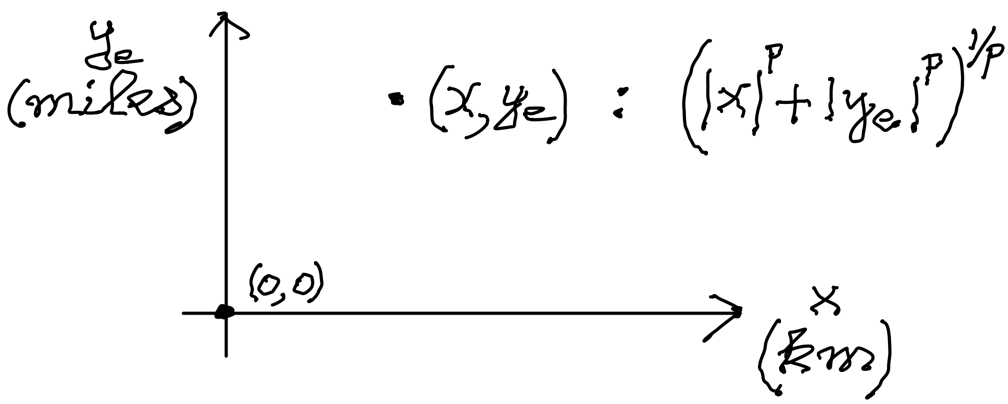


Figure 3.1: Incommensurable coordinate functions

Then there are many options for defining the separation between points, one of them being

$$\mathrm{sep}\big((0,0);(x,y_e)\big) \equiv \left(|x|^p + |y_e|^p\right)^{1/p}, \quad p \geq 1$$

and that is very useful in mathematics and theoretical engineering in spite of the fact that x and y_e have different units.

However, in order to define the concept "distance", the units along the two coordinate axes must be commensurable. Letting $k = .625$ km/mile, one recoordinates the domain using commensurable coordinate functions:

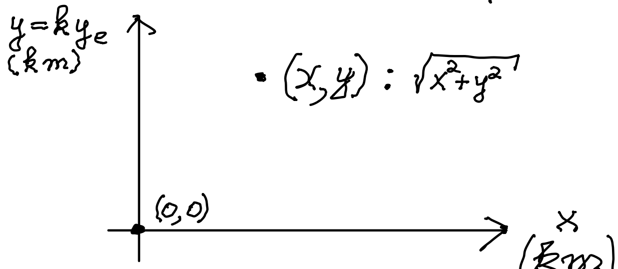


Figure 3.2: Commensurable coordinate functions

II. Euclidean space vs. Lorentzian spacetime.

The "distance" between two points is defined as the familiar Pythagorean distance:

$$d\left((0,0);(x,y)\right) \equiv \left(x^2+y^2\right)^{1/2}.$$

With such a coordinate system, reorienting the coordinate axes is called a rotation.

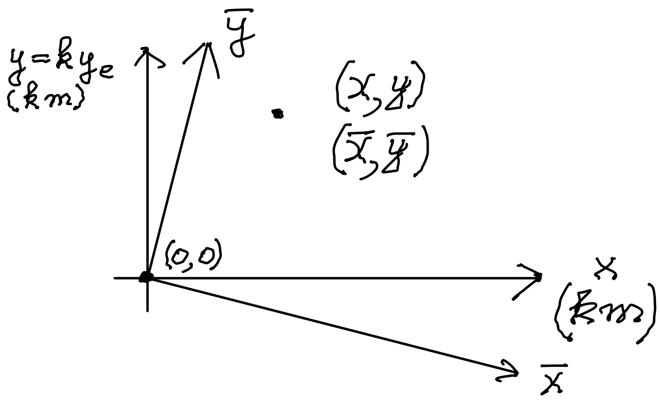


Figure 3.3: Rotation of a coordinate system.

In the Euclidean plane one finds that the (squared) "distance" between a pair of points is invariant under rotation:

$$\boxed{x^2+y^2=(\bar{x})^2+(\bar{y})^2.}$$

The invariance as expressed in this box holds whenever the the coordinates are commensurate.

The principle of the invariance of distance for Euclidean space extends to invariance of the interval for Lorentz spacetime.

Example 1: Pair of Frames

a) In Euclidean space two different coordinate system can be used to determine the distance between any pair of points

$$(\Delta x)^2+(\Delta y)^2=(\Delta\bar{x})^2+(\Delta\bar{y})^2\equiv(\Delta\sigma)^2$$

If one uses incommensurate units along different axes, one must introduce the appropriate conversion factor, say k, in order to make them commensurate. The result is that

$$(\Delta\sigma)^2=(\Delta\bar{x})^2+(k\,\Delta\bar{y}_e)^2.$$

The unbarred coordinate differences are related to the barred ones by means of a Euclidean rotation.

b) In spacetime either of two different frames can be used to determine the interval, provided one uses relativistic time difference Δt instead of conventional time difference,

$$\begin{aligned}\Delta t&=c\,\Delta t_{conv}\quad[\text{length}]\\ \Delta\bar{t}&=c\,\Delta\bar{t}_{conv}\quad[\text{length}]\end{aligned}\quad\text{same}$$

In terms of these, the interval between two given (point-) events is

3.5

$$\left.\begin{aligned}(c\Delta t_{conv})^2-(\Delta x)^2&=(c\Delta\bar{t}_{conv})^2-(\Delta\bar{x})^2\\(\Delta t)^2-(\Delta x)^2&=(\Delta\bar{t})^2-(\Delta\bar{x})^2\end{aligned}\right\}\equiv(\Delta\tau)^2$$

The respective coordinate separations are in general different in the two frames.

$$\Delta x\neq\Delta\bar{x} \text{ and } \Delta t\neq\Delta\bar{t}.$$

However, they are related, as we shall see by a <u>Lorentz rotation.</u>

[A quick check miniproblem: Is it true that $(\Delta t)^2-(\Delta x)^2=0 \Leftrightarrow (\Delta\bar{t})^2-(\Delta\bar{x})^2=0$? If so, why?]

Example 2: Many frames

a) Let the position of the point in Euclidean space be measured by many surveyors that are rotated relative to one another around the origin

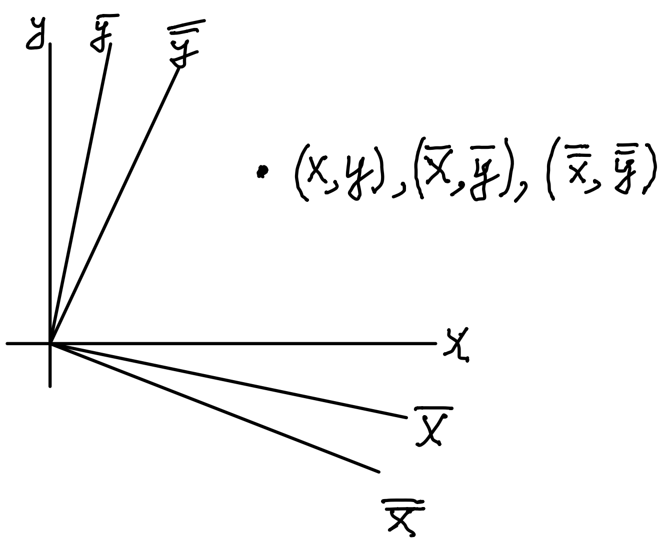


Figure 3.4: Several rotated frames in Euclidean space.

Plotting their measurements on a graph one obtains

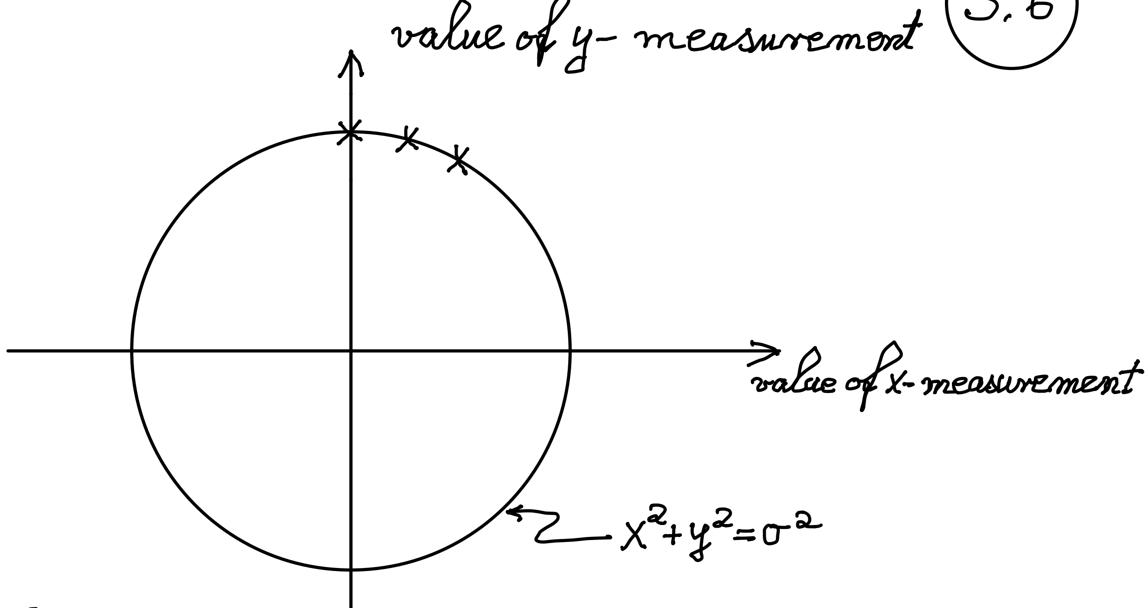


Figure 3.5: Data from measurements of a pair of points in the Euclidean plane

One obtains a circle mathematized by $x^2+y^2=\sigma^2$

b) Similarly consider many inertial spacetime observers, all moving in their common x-direction relative to one another. Let them observe a pair of events, one at $t=\bar{t}=\bar{\bar{t}}=\cdots=0$, $x=\bar{x}=\bar{\bar{x}}=\cdots=0$, the other at $(\Delta x,\Delta t)$, $(\Delta\bar{x},\Delta\bar{t})$, $(\Delta\bar{\bar{x}},\Delta\bar{\bar{t}})$, ...

Invariance of the interval implies

$$(\Delta t)^2-(\Delta x)^2=(\Delta\bar{t})^2-(\Delta\bar{x})^2=(\Delta\bar{\bar{t}})^2-(\Delta\bar{\bar{x}})^2=\cdots=(\Delta\tau)^2$$

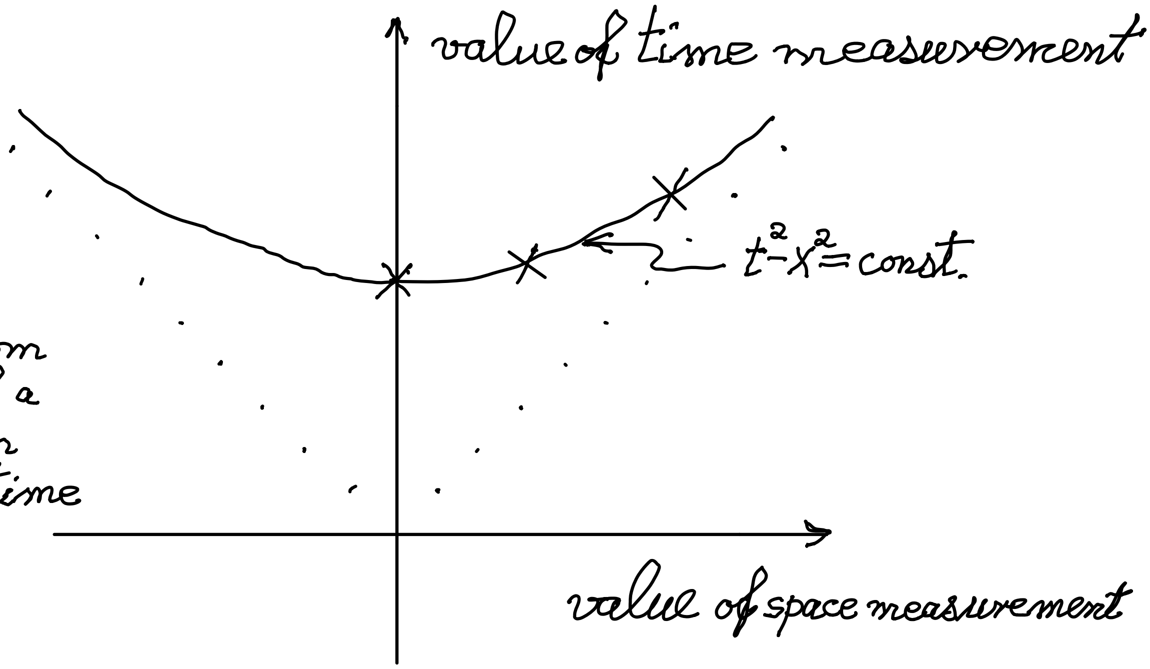


Figure 3.6: Data from measurements of a pair of events in Lorentz spacetime

The result is a hyperbola mathematized by $t^2-x^2=\tau^2$.

Example 3: Perpendicularity in Euclidean space corresponds to Simultaneity in Lorentz spacetime

a) Given two points on a straight line in Euclidean space, the locus of points equidistant from the two point is an axis perpendicular to the given line

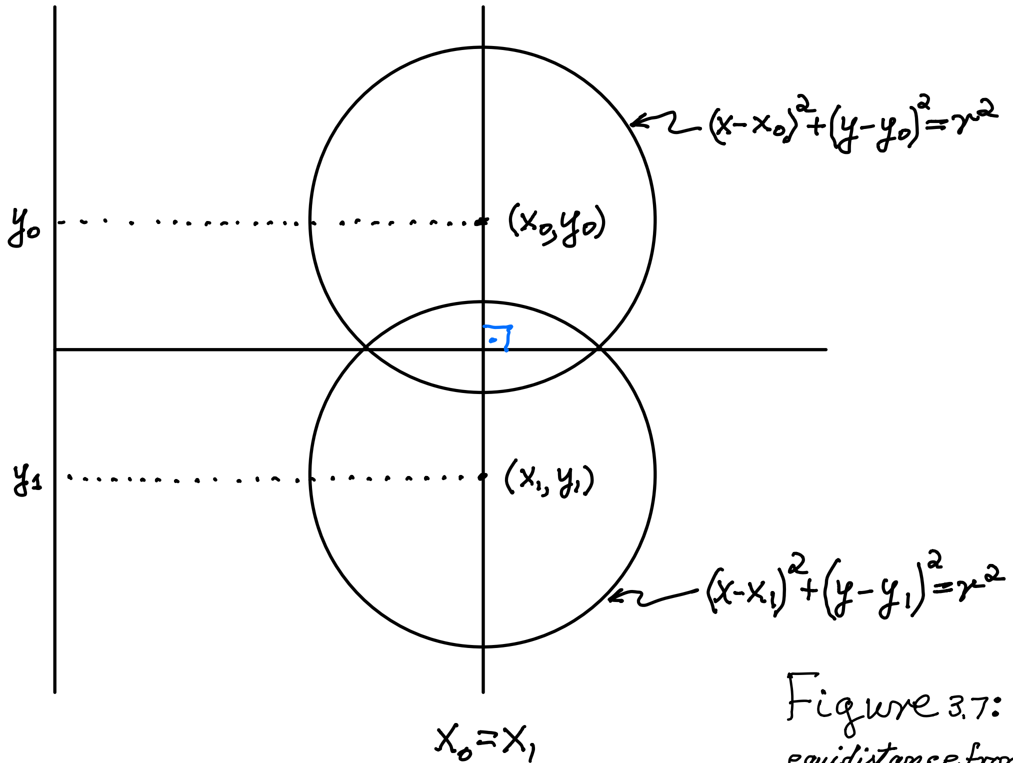


Figure 3.7: Perpendicularity via equidistance from two points (x_0,y_0) and (x_1,y_1) on the given line.

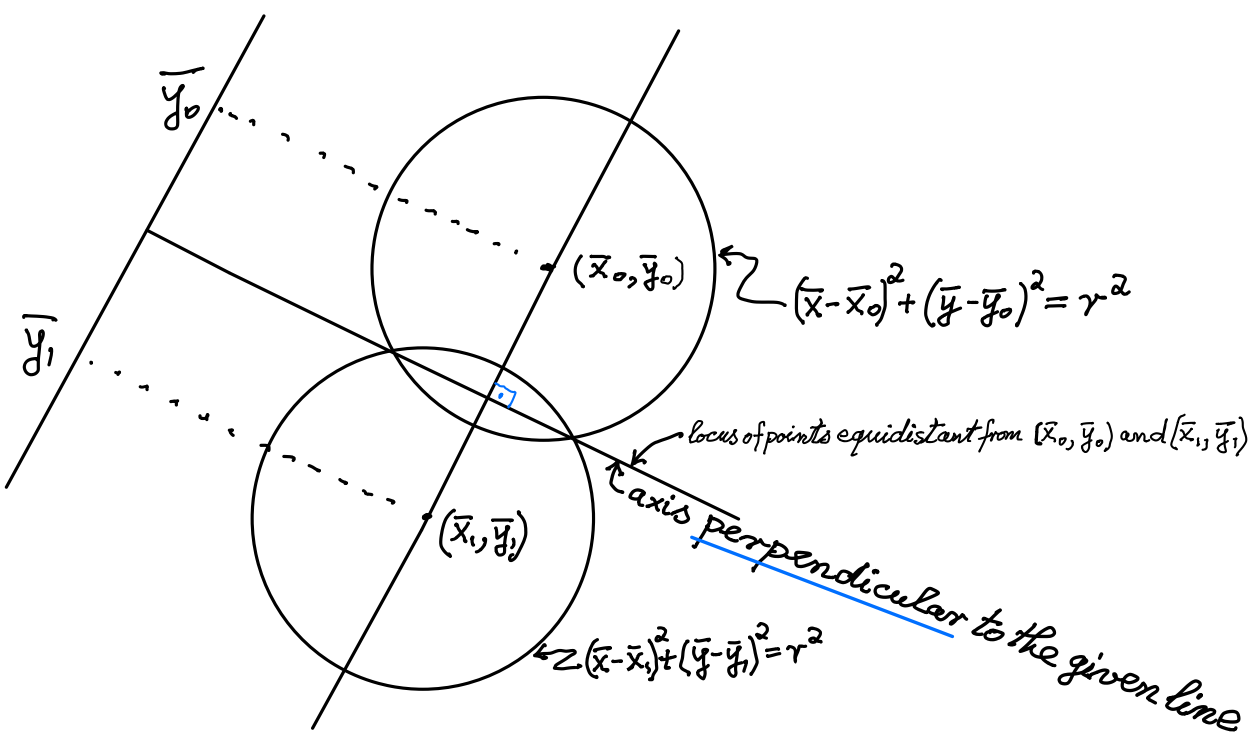


Figure 3.8: Perpendicularity via equidistance from two points $(\bar{x}_0, \bar{y}_0)$ and $(\bar{x}_1, \bar{y}_1)$ on a given rotated line.

3.8

b) By extending the geometrical perpendicularity in Euclidean space to that in Lorentz spacetime one arrived at the geometrization of the concept of simulteneity.

Given two consecutive events on the straight worldline of a spacetime observer; a set of events is said to be <u>simultaneous relative to this observer worldline</u> if each of them has equal interval separation from the two given events.

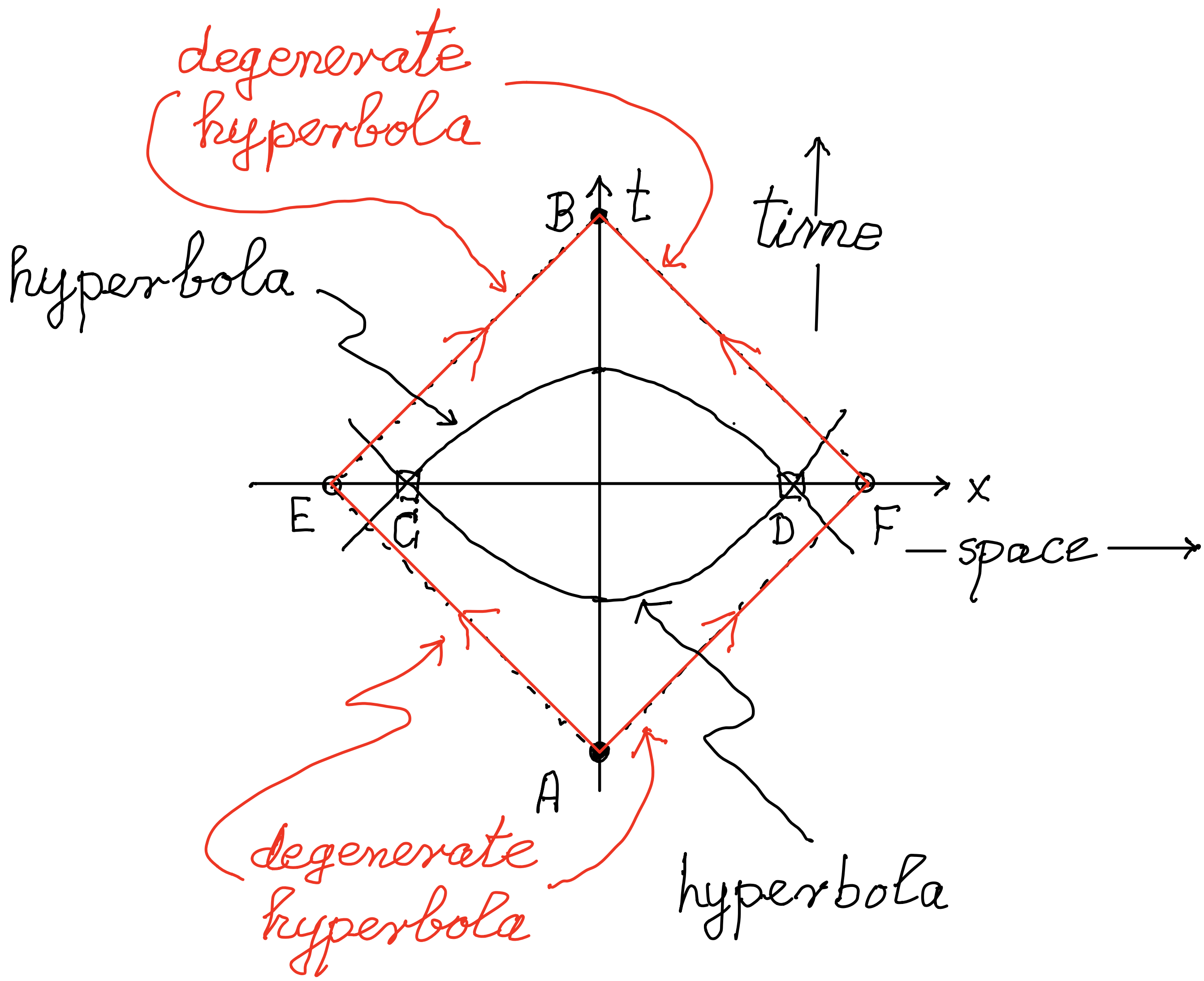


Figure 3.9: Simultaneity via equal interval separation. 3.9

Events E, C, D, and F are simultaneous relative to the worldline of observer $\overline{AB}$. This is because events A and B have equal interval separation from each of the events E, C, D and F. If two radar pulses emitted from A get reflected from E and F so as to be received in coincidence at B, then E and F are said to be simultaneous relative to $\overline{AC}$.

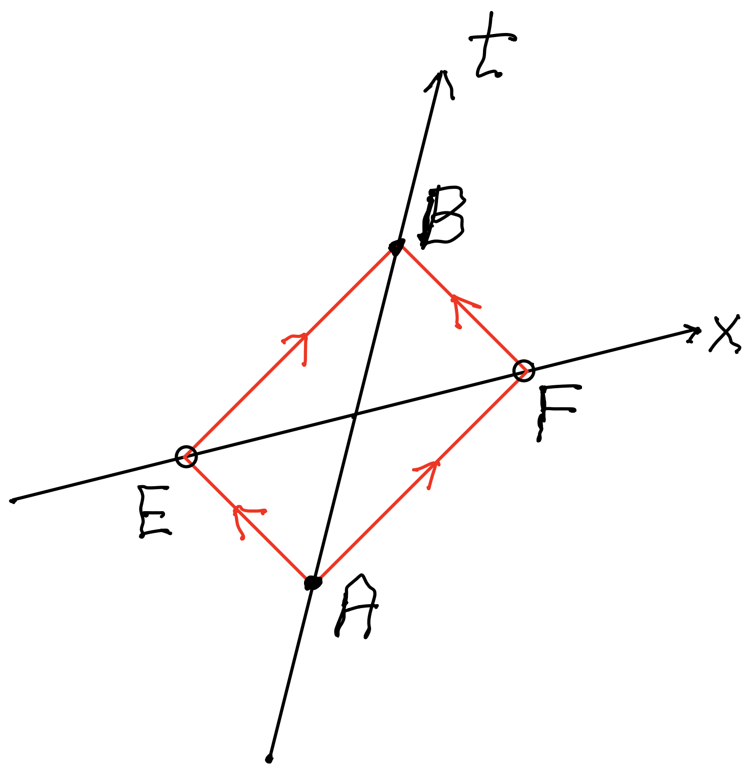
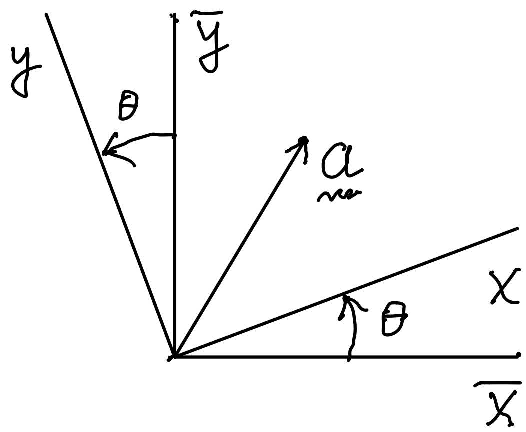


Figure 3.10: Events E and F are simultaneous relative to $\overline{AC}$ but not relative to a frame in motion with respect to $\overline{AC}$.

Example 4: Transformation between a pair of frames

a) In Euclidean space

(i) Recall the rotation transformation between a pair of Euclidean frames



COORD. COMP.
RELATIVE TO
OLD BASIS

$$\left.\begin{aligned} x &= \bar{x}\cos\theta + \bar{y}\sin\theta \\ y &= -\bar{x}\sin\theta + \bar{y}\cos\theta \end{aligned}\right\} \begin{pmatrix} x \\ y \end{pmatrix} = T_\theta \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \quad \text{or} \quad (\{a\}) = T_\theta (\{\bar{a}\}) \qquad (3.1)$$

COORD. COMP.
RELATIVE TO
NEW BASIS

This transformation is

(a) linear : WHY?

(b) has trigonometric coefficients : WHY?

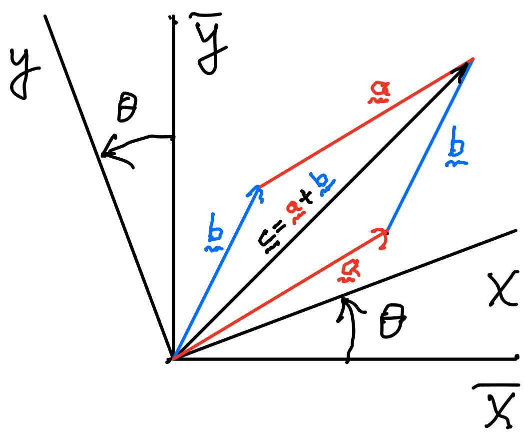
Answer: (a) Homogeneity of Euclian space

(b) $x^2+y^2 = \text{const}$

(a) Homogeneity, i.e. sameness under translation, implies that for a a triangle formed by the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, and $\underset{\sim}{c}$: $\underset{\sim}{a}+\underset{\sim}{b}=\underset{\sim}{c}$, with respective coordinate representatives

$$\underset{\sim}{a}:\quad \{a\} = T_\theta(\{\bar{a}\})$$
$$\underset{\sim}{b}:\quad \{b\} = T_\theta(\{\bar{b}\})$$
$$\underset{\sim}{c}:\quad \{c\} = T_\theta(\{\bar{c}\})$$

NEW OLD coordinate representatives



Homogeneity, i.e. equivalence under parallel transport, guarantes that if the vectors $(\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c})$ form a triangle, i.e. $\underset{\sim}{a}+\underset{\sim}{b}=\underset{\sim}{c}$, then so do their coordinate representatives

$$\{c\} = \{a\} + \{b\} \qquad \text{NEW}$$
$$\{\bar{c}\} = \{\bar{a}\} + \{\bar{b}\} \qquad \text{OLD}.$$

Apply the coordinate transformation and obtain

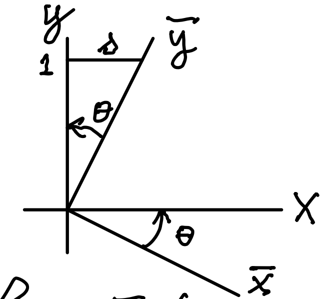
$$\boxed{\underbrace{T_\theta\big(\overbrace{\{\bar{a}\}+\{\bar{b}\}}^{\{\bar{c}\}}\big)}_{\{c\}} = \underbrace{T_\theta(\{\bar{a}\})}_{\{a\}} + \underbrace{T_\theta(\{\bar{b}\})}_{\{b\}}}$$

Thus T_θ is a linear transformation indeed.

(b) The linear transformation T_θ has trigonometric coefficients ($\cos\theta, \sin\theta$). This is necessary in order to guarantee the invariance of the squared distance

$$x^2+y^2=\bar{x}^2+\bar{y}^2.$$

(ii) Instead of using θ to parametrize T_θ, use the slope



$$s=\frac{\sin\theta}{\cos\theta}=\tan\theta$$

to characterize the transformation law Eq.(3.1) on page 3.10:

$$x=\frac{\bar{x}+s\bar{y}}{\sqrt{1+s^2}} \quad (=\bar{x}\cos\theta+\bar{y}\sin\theta)$$

$$y=\frac{-s\bar{x}+\bar{y}}{\sqrt{1+s^2}} \quad (=-\bar{x}\sin\theta+\bar{y}\cos\theta)$$

(b) In Lorentz spacetime

introduce a transformation law with the same requirements

(i) Linearity (because of the P. of R. plus invariance of spacetime parallelism under change of frames)

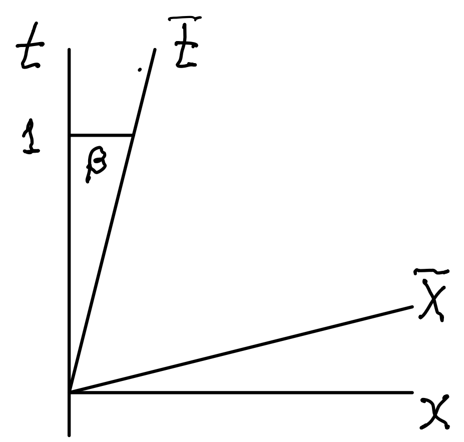
(ii) Invariance of the interval. 3.13

Thus the transformation between inertial frames is

$$\left.\begin{aligned} t &= \bar{t}\cosh\theta + \bar{x}\sinh\theta \\ x &= \bar{t}\sinh\theta + \bar{x}\cosh\theta \end{aligned}\right\} \begin{pmatrix} t \\ x \end{pmatrix} = \Lambda \begin{pmatrix} \bar{t} \\ \bar{x} \end{pmatrix}$$

If instead of the "rapidity" parameter θ, one uses the slope

$$\beta = \frac{\sinh\theta}{\cosh\theta} = \tanh\theta$$



then

$$t = \frac{\bar{t} + \beta\bar{x}}{\sqrt{1-\beta^2}}$$

$$x = \frac{\beta\bar{t} + \bar{x}}{\sqrt{1-\beta^2}}$$

Here the slope is simply the velocity between the two frames. This is because $\bar{x}=0$ yields the velocity of the spatial origin of $\bar{S}$:

$$\text{velocity} = \frac{x}{t} = \frac{\sinh\theta}{\cosh\theta} = \beta$$

This transformation of the set of coordinates relative to one frame to that relative to another inertial frame is called a <u>Lorentz transformation</u>.