LECTURE 31

I. Commutator in the Euclidean Plane

II. Covariant derivative of a vector

III. Covariant derivative of a covector

IV. Commutator vs covariant derivative

In MTW read Section 14.5; Sections 8.3, 8.5, 10.3, 10.4
Box 10.2, 10.3

I. Commutator of vector fields in the Euclidean plane.

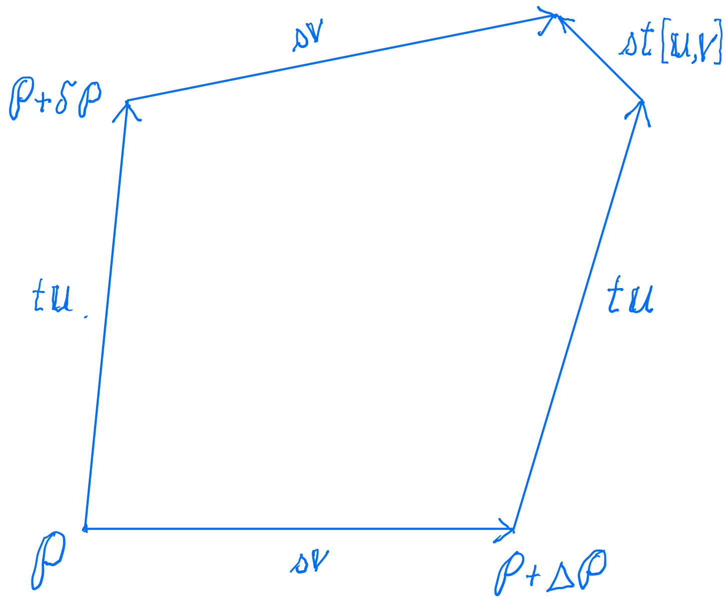


Figure 31.1: The commutator of two vector fields at a point is the vector necessary to close the quadrilateral to be formed from the joined curve segments tangent to the vectors in the local neighborhood of that point.

Compare and contrast the two commutators formed from two different pairs of vector fields:
One formed from the pair of polar coordinate-induced basis vectors

$$\{e_r, e_\theta\} \equiv \left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$$

assigned to each point of the Euclidean plane, and their commutator is

$$[e_r, e_\theta] = 0 \ ;$$

the other formed from the pair of normalized physical basis vectors

$$\{e_{\hat{r}}, e_{\hat{\theta}}\} \equiv \left\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\right\}$$

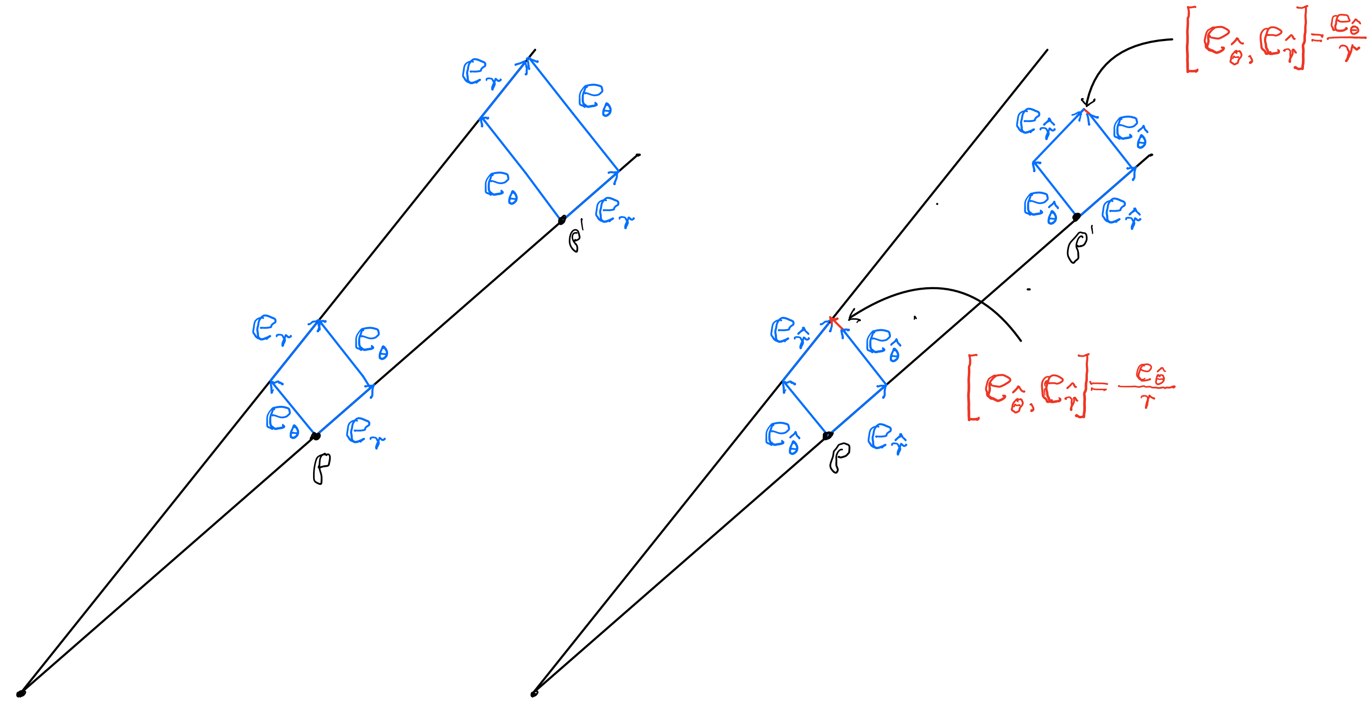
assigned to each point of the same Euclidean plane, and their commutator is

$$[e_{\hat\theta}, e_{\hat r}] = \frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r} - \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\frac{\partial}{\partial r} - \frac{-1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial\theta} .$$

$$= \frac{1}{r}\, e_{\hat\theta}$$

The geometrical difference between these two commutators is depicted in Figure 31.2



Coordinate basis vector fields

Normalized basis vector fields

Figure 31.2: Commutator for two different pairs of vector fields in the polar coordinatized Euclidean plane.

II. Covariant derivative of vector field $\mathbf{v} = v^i \mathbf{e}_i$

Relative to a coordinate-induced basis $\{\mathbf{e}_i = \frac{\partial}{\partial x^i}\}$

$$\nabla_{\mathbf{e}_k} \mathbf{e}_i \equiv \langle d\mathbf{e}_i \,|\, \mathbf{e}_k \rangle = \langle \mathbf{e}_j \, \underset{\sim}{\omega}^j{}_i \,|\, \mathbf{e}_k \rangle$$

$$= \mathbf{e}_j \langle \Gamma^j{}_{i\ell} \, dx^\ell \,|\, \frac{\partial}{\partial x^k} \rangle$$

$$\boxed{\nabla_{\mathbf{e}_k} \mathbf{e}_i = \mathbf{e}_j \Gamma^j{}_{ik}} \qquad (31.1)$$

Using defining properties of the covariant derivative one finds

$$\nabla_{\mathbf{u}} \mathbf{v} = \nabla_{u^k \mathbf{e}_k} (v^i \mathbf{e}_i)$$

$$= u^k \left[\nabla_{\mathbf{e}_k} (v^i) \mathbf{e}_i + v^i \nabla_{\mathbf{e}_k} (\mathbf{e}_i) \right]$$

$$= \left[\frac{\partial v^i}{\partial x^k} \mathbf{e}_i + v^i \mathbf{e}_j \Gamma^j{}_{ik} \right] u^k$$

$$\boxed{\nabla_{\mathbf{u}} \mathbf{v} = \mathbf{e}_j \left[\frac{\partial v^j}{\partial x^k} + v^i \Gamma^j{}_{ik} \right] u^k}$$

$$\underbrace{\frac{\partial v^j}{\partial x^k} + v^i \Gamma^j{}_{ik}}_{v^j{}_{;k}}$$

The coefficients $v^j{}_{;k}$ are the <u>components</u> of $\nabla_{\mathbf{u}} \mathbf{v}$.

III. Covariant derivative of a 1-form $\underset{\sim}{\sigma} = \sigma_i \underset{\sim}{\omega}^i$

The parallel transport of covectors, in particular the mathematization of the deviation of a covector from parallelism, is known once that of the basis elements $\{\omega^i\}$ is known.

It is a fact that a basis $\{\mathbf{e}_i\}$ and its dual basis $\{\underset{\sim}{\omega}^j\}$ exist in every tangent space throughout the manifold, i.e. that the condition

$$\langle \underset{\sim}{\omega}^j \,|\, \mathbf{e}_i \rangle = \delta^j{}_i$$

holds at every point of the manifold. To negate this condition would be to negate the existence of a vector space.

A) To guarantee that the law of parallel transport of vectors (defined on page 30.8) does not violate this condition, one must have

$$0=\nabla_{e_k}(\delta^i{}_j)=\nabla_{e_k}\langle\omega^i|e_j\rangle$$
$$=\langle\nabla_{e_k}\omega^i|e_j\rangle+\langle\omega^i|\nabla_{e_k}e_j\rangle$$

Consequently, using $\nabla_{e_k}e_j=e_\ell\Gamma^\ell{}_{jk}$, Eq. (31.1) on page 31.4, one finds

$$\boxed{\langle\nabla_{e_k}\omega^i|e_j\rangle=-\Gamma^i{}_{jk}}$$

These are the expansion coefficientis of the covectors $\nabla_{e_k}\underset{\sim}{\omega}^i$. Indeed, using the spanning property of the basis elements (i.e. $\underset{\sim}{f}=\langle f|e_j\rangle\underset{\sim}{\omega}^j$; see Lecture 12, page 12.8), one finds that

$$\nabla_{e_k}\underset{\sim}{\omega}^i=\langle\nabla_{e_k}\underset{\sim}{\omega}^i|e_j\rangle\,\omega^j$$
$$\boxed{\nabla_{e_k}\underset{\sim}{\omega}^i=-\Gamma^i{}_{jk}\,\underset{\sim}{\omega}^j}$$

B) Extend this result to a general 1-form $\underset{\sim}{\sigma}=\sigma_i\underset{\sim}{\omega}^i$ by using the <u>product rule</u> for differentiation (For unit-economy we from now on write $\nabla_{e_k}[\cdots]\equiv\nabla_k[\cdots]$):

$$\nabla_k(\sigma_i\underset{\sim}{\omega}^i)=(\nabla_k\sigma_i)\underset{\sim}{\omega}^i+\sigma_i\nabla_k\underset{\sim}{\omega}^i$$
$$=\frac{\partial\sigma_i}{\partial x^k}\underset{\sim}{\omega}^i-\sigma_j\Gamma^j{}_{ik}\underset{\sim}{\omega}^i \qquad e_k=\frac{\partial}{\partial x^k}$$

The covariant derivative of $\underset{\sim}{\sigma}$ is therefore

$$\boxed{\nabla_k\underset{\sim}{\sigma}=\nabla_k(\sigma_i\underset{\sim}{\omega}^i)=\left(\frac{\partial\sigma_i}{\partial x^k}-\sigma_j\Gamma^j{}_{ik}\right)\omega^k}$$

where

$$\boxed{\frac{\partial\sigma_i}{\partial x^k}-\sigma_j\Gamma^j{}_{ik}\equiv\sigma_{i;k}}$$

are the components of $\nabla_k\underset{\sim}{\sigma}$. They are to be compared with

$$v^i{}_{;k}=\frac{\partial v^i}{\partial x^k}+v^j\Gamma^i{}_{jk}$$

for $v=e_i v^i$.

IV. Commutator vs. Covariant derivative

Consider two vector fields u and v on manifold M with a law of parallel transport.

Consider the vectors u along the curves $c_v(s;x^k)$ and $c_v(s;x^k+\Delta x^k)$ of the flow field of v

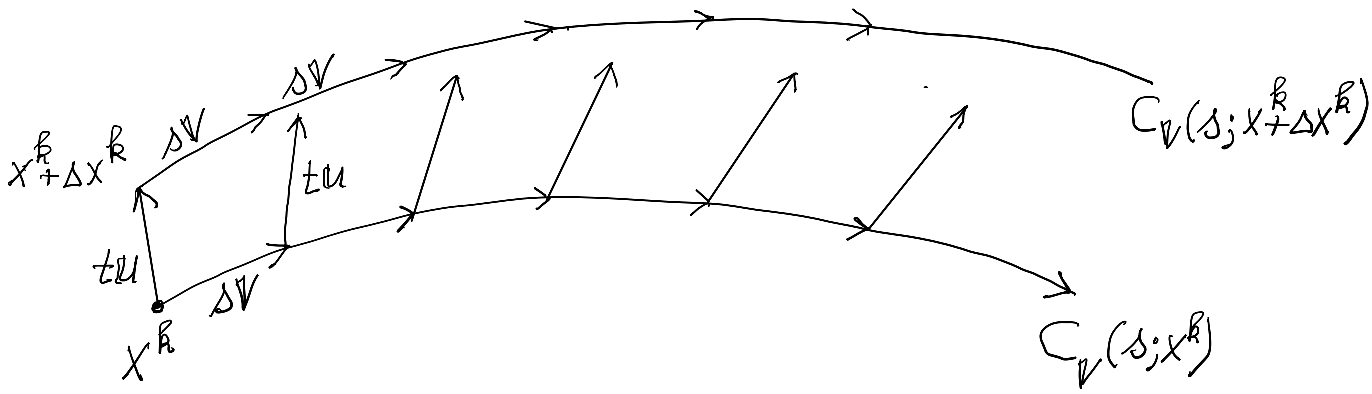


Figure 31.3: Two curves, $c_v(s;x^k)$ and $c_v(s;x^k+\Delta x^k)$, of the flow field v. Their starting points are $\{c_v^i(s=0;x^k)=x^i\}_{i=1}^n$ and $\{c_v^i(s=0;x^k+\Delta x^k)=x^i+\Delta x^i\}_{i=1}^n$.

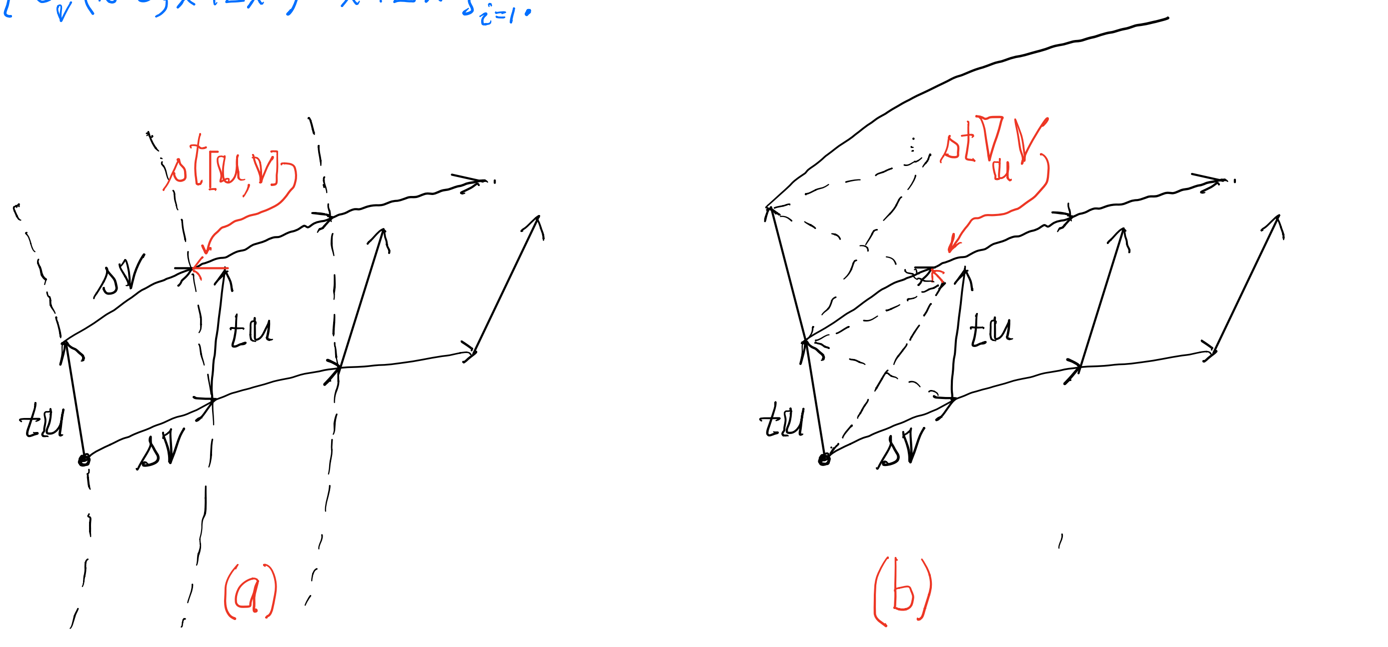


Figure 31.3: (a): v-flow induced mapping $st[u,v]$. (b): Parallel transport-induced

mapping st $\nabla_u V$.

A) Given the vector $u \in T_p(M)$, one finds

$$\nabla V \; (= dV): \quad T_p(M) \longrightarrow T_p(M)$$
$$u \rightsquigarrow \nabla_u V \; (= \langle dV | u \rangle)$$

is a <u>tensor map</u> because ∇V is <u>pointwise</u> linear:

$$f u_1 + g u_2 \rightsquigarrow f \nabla_{u_1} V + g \nabla_{u_2} V$$

Thus <u>∇V is a tensor</u> at point p

B) By contrast, consider the "Lie derivative" $\mathcal{L}_V$ of u:

$$\mathcal{L}_V : \quad T_p(M) \longrightarrow T_p(M)$$
$$u \rightsquigarrow \mathcal{L}_V u = [V, u]$$

This is not a tensor map, because for

$$\begin{aligned} fu \rightsquigarrow \mathcal{L}_V(fu) &= [V, fu] = V fu - fuV \\ &= f[V,u] + V(f)u \\ &= f \mathcal{L}_V(u) + u(f)V \end{aligned}$$

as well as for

$$\begin{aligned} gV \rightsquigarrow \mathcal{L}_{gV} u &= [gV, u] = gVu - ugV \\ &= g[V,u] - u(g)V \\ &= g\mathcal{L}_u V - V(g)\, u \end{aligned}$$

$u(f) \neq 0$ and $V(g) \neq 0$, which spoil the point-wise linearity