LECTURE 32

I. Parallel Vector Fields

II. The Torsion Tensor as an imprint of Parallelism on a Manifold

In MTW read Box 10.2 (Note: In the middle of their page 260 there is a typo: "chain rule" should have read "product rule" instead. Strangely enough, the same typo can be seen in their Exercise 10.2, 10.3, 10.4, and 10.5. The chain rule applies to the differentiation of the composition of two functions, not to their product.)

I. Parallel Vector Fields

As soon as one knows the covariant derivative $\nabla_k \mathbf{e}_i = \mathbf{e}_j \Gamma^j{}_{ik}$, or equivalently $d\mathbf{e}_i = \mathbf{e}_j \otimes \omega^j{}_i$, one can construct parallel vector fields. Such vector fields differ from others, say $\mathbf{v}$, which are non-parallel along a given direction $\mathbf{u}$, i.e. those for which

$$\nabla_{\mathbf{u}} \mathbf{v} \neq 0.$$

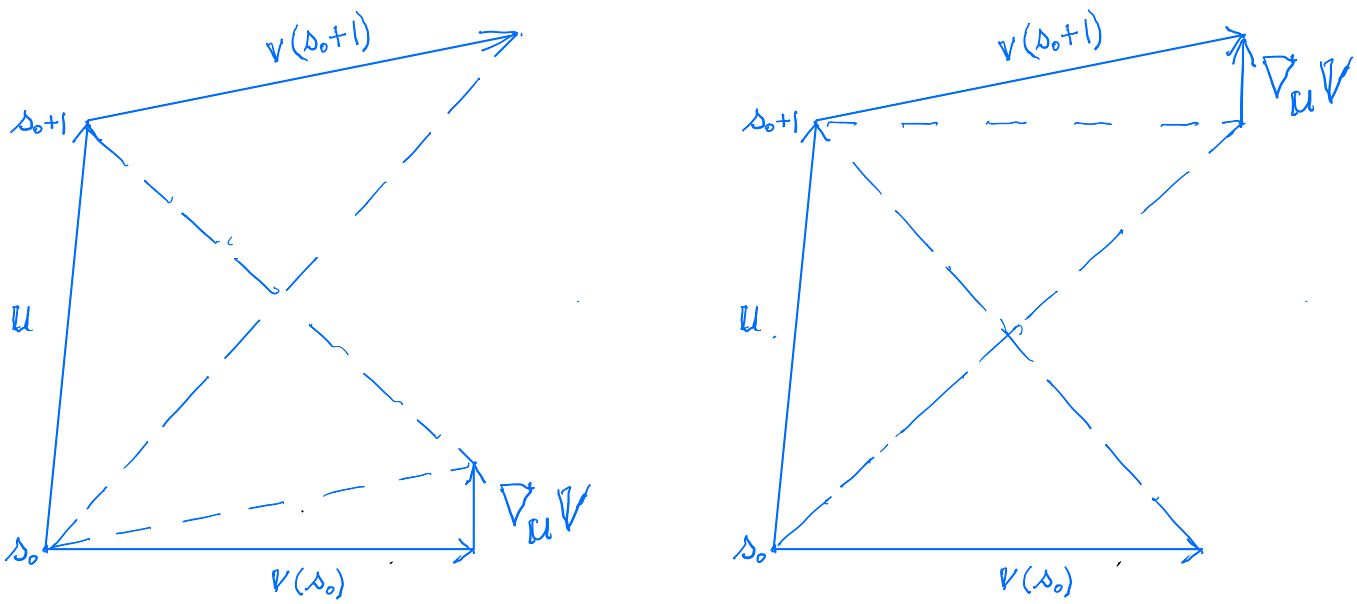

Figure 32.1: The vector $\nabla_{\mathbf{u}} \mathbf{v}$ is the vectorial deviation from parallelism between neighboring vectors $\mathbf{v}(s_0)$ and $\mathbf{v}(s_0+1)$ of the given non-parllel vector field $\mathbf{v}$.

However, if a vector field, say

$$\mathbf{Y} = \mathbf{e}_i y^i,$$

<u>is</u> parallel under transport along the direction

$$\mathbf{u} = \mathbf{e}_k u^k,$$

then

$$\boxed{\nabla_{\mathbf{u}} \mathbf{Y} = 0.}$$

This equation is a compact way stating the problem of solving a system of ordinary diffential equations as follows:

GIVEN: a) $\mathbb{u}=\mathbb{e}_k u^k(x)$, where the $\mathbb{e}_k=\frac{\partial}{\partial x^k}$'s are the coordinate basis vectors.

b) One of $\mathbb{u}$'s integral curves

$$c(t):\{c^k(t)\}$$

$$\frac{dc}{dt}=\mathbb{u}:\left\{\frac{dc^k}{dt}=u^k(c(t))\equiv u^k(t)\right\}$$

SOLVE: $\nabla_{\mathbb{u}}\mathbb{Y}=0$ for $\mathbb{Y}(t)$

where $\mathbb{u}=\mathbb{e}_k u^k(t)$

and

$$\mathbb{Y}(t=t_0)=\mathbb{V}_0 : y^i(t=t_0)=v_0^i$$

is the initial vector at $c^k(t_0)$ of the given curve $c^k(t)$.

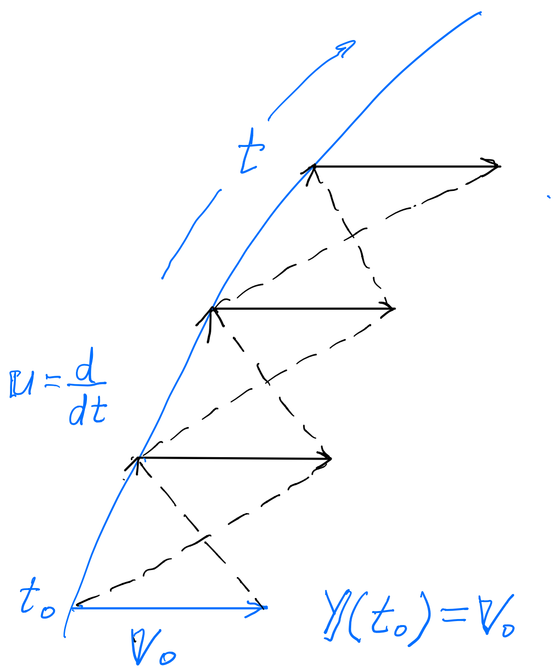

Figure 32.2: Vector field $\mathbb{Y}(t)$ parallel (colored in black) to $\mathbb{v}_0$ along the given curve $c(t)$ whose tangent is $\mathbb{u}=\frac{dc}{dt}$.

SOLUTION: $0 = \nabla_{\mathbf{u}} \mathbb{Y} = \mathbf{e}_i \left(u^k \frac{\partial y^i}{\partial x^k} + y^j \Gamma^i{}_{jk} u^k \right)$ refers to the following system of linear coupled o.d.e.'s for $y^i(t)$:

$$0 = \frac{dy^i}{dt} + y^j \underbrace{\Gamma^i{}_{jk}\left(c^\ell(t)\right) u^k(t)}_{\text{given}} .$$

From the existence and uniqueness theorem of Lecture 25 one knows that this system has a unique solution $\{y^i(t)\}$ for the specified initial condition $y^i(t=t_0) = v^i_0$

The solution $\mathbb{Y}(t) = \mathbf{e}_i\, y^i(t)$ to the differential equation $\nabla_{\mathbf{u}} \mathbb{Y} = 0$ is a vector field along the given curve $\{c^k(t)\}$ whose tangent is $\{u^k = \frac{dc^k}{dt}\}$, namely

$$\mathbb{Y}(t) = y^i(t) \frac{\partial}{\partial x^i} ,$$

which satisfies

$$0 = \nabla_{\mathbf{u}} \mathbb{Y} = \mathbf{e}_i \left(\frac{dy^i}{dt} + y^j \Gamma^i{}_{jk}\left(c^\ell(t)\right) u^k \right).$$

II. Cartan's Torsion Tensor

Parallelism is a system for comparing vectors at different points of a manifold. Parallelism is determined by specifying a set of connection 1-forms $\underset{\sim}{\omega}^i{}_j = \Gamma^i{}_{jk}\underset{\sim}{\omega}^k$ on a manifold. They, by virtue of

$$de_i = e_j \underset{\sim}{\omega}^j{}_i,$$

are the means for mathematizing this comparison.

Parallelism in the Euclidean plane manifests itself in the form of closed parallelograms.
However, in a crystal with srew or edge dislocations between parallel crystal planes, parallelism manifest itself in the form of parallelograms which are open. See Figures 32.3 and 32.6 below.

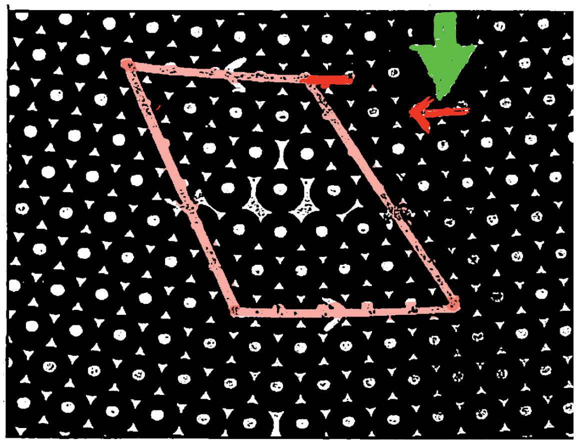

Figure 32.3: Parallelism with non-zero torsion. The red sides of the rows of disks are pair-wise parallel, but the parallelogram is not closed: the dark red segment connecting the two disk centers in the upper right is the (negative) gap between the upper and the right-hand side of the parallelogram. This gap is a displacement vector.
It is the red Burger vector which is depicted in isolation below the green indicator above it.
The cause of this vectorial gap is the crystal defect at the center of the parallelogram. The resulting non-zero torsion is the vectorial displacement per area enclosed by the parallelogram.

Parallelism under which parallelograms are not closed manifests itself as a non-zero torsion tensor field on the manifold.

Let u and v be two smooth vector fields whose is $[u,v]$ and whose covariant derivatives are $\nabla_u v$ and $\nabla_v u$.
Whenever parallelism manifest itself in the form

of closed parallelograms, one infers from the defining diagrams in Figures 31.1 and 32.1 that the triangle formed by $\nabla_u v$, $\nabla_v u$, and $[u,v]$ is closed:

$$\nabla_u v - \nabla_v u - [u,v] \equiv \mathbb{T}(u,v) = 0 \qquad (32.1)$$

The vanishing of their linear combination is depicted in Figure 32.4

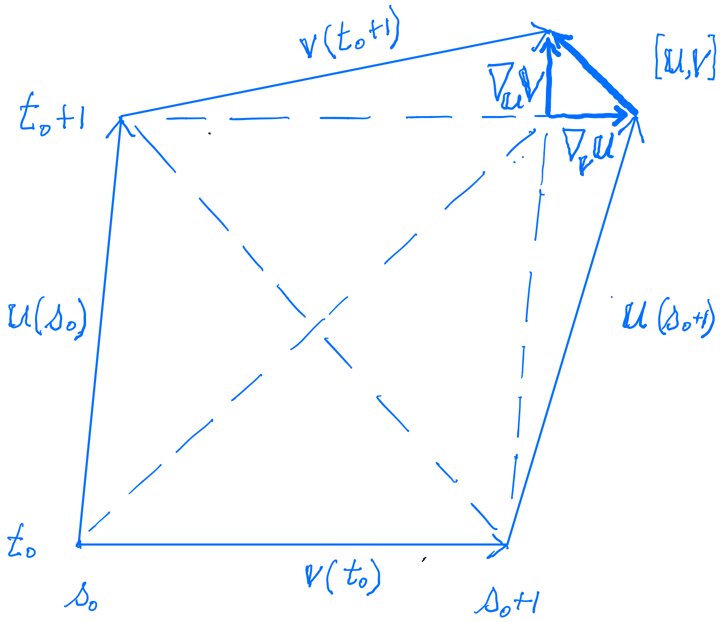

Figure 32.4: Parallelism with closed parallelograms is characterized by $\nabla_u v - \nabla_v u - [u,v] = 0$.

By contrast, parallelism with open parallelograms is characterized by a non-zero displacement vector, the Burger vector,

$$\nabla_u v - \nabla_v u - [u,v] \equiv \mathbb{T}(u,v),$$

which fills the non-zero gap of an open parallelogram. This is depicted in Figure 32.5.

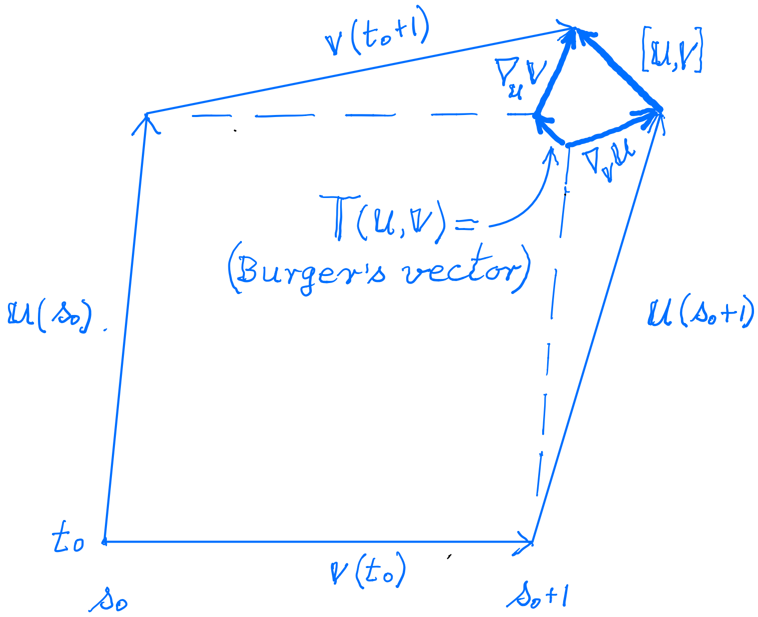

Figure 32.5: For parallelism which admits parallelograms with a gap. This gap is a non-zero displacement vector, the Burger vector, which is the vectorial value of $\mathbb{T}(u,v)$.

Every system of parallelism on a manifold leaves a geometrical footprint in the form of that system's torsion tensor. This fact is mathematized by the following

Propoposition ("Cartan's Torsion Tensor")

The pointwise linear map

$$\mathbb{T}: \quad T_p(M) \times T_p(M) \longrightarrow T_p(M)$$

$$(u, v) \rightsquigarrow \mathbb{T}(u,v) = \nabla_u v - \nabla_v u - [u,v]$$

has the property

$$\mathbb{T}(fu, v) = f\,\mathbb{T}(u,v)$$

$$\mathbb{T}(u, gv) = g\,\mathbb{T}(u,v)$$

at each point $\mathcal{P}$ for all $f, g \in C^\infty(\mathcal{P}, M, R)$.
Thus $\mathbb{T}$ is a tensor at each point $\mathcal{P}$.

Comment

A system of parallel transport is said to be consistent or integrable whenever its torsion tensor $\mathbb{T}$ vanishes, i.e.

$$\mathbb{T}(u, v) = 0$$

for all vector fields u and v.

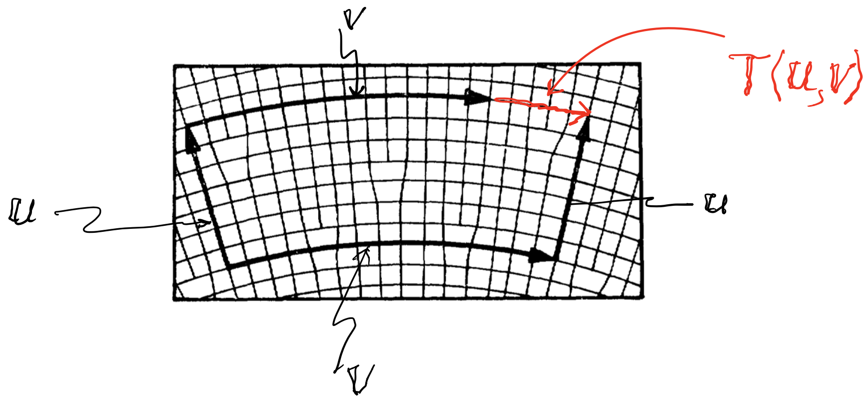

Figure 32.6: A parallelogram in a crystal permeated by a distribution of dislocations has a shape which differs from that of a Euclidean parallelogram without dislocations inside of it. The physical cause of the shape difference is the existence of dislocations (here edge dislocations, five of them) in the interior of the parallelogram.
The quantitative effect is in the form of the (red) Burger vector whose size is five horizontal lattice vectors.

The vector-valued dislocation density is mathematized by the torsion tensor in 3-d, i.e. torsion and dislocation density are proportional to each other.

The response of the crystal to the existence of a non-zero dislocation density is that it produces a moment of stress that distorts the originally Euclidean shape of the crystal. This is depicted in Figure 32.6.

The spacetime generalization of that moment of stress is the intrinsic spin angular momentum tensor [see Box 5.6 in MTW]. The spin density in matter is an example of this.

The torsion tensor is a concept that applies to any number of dimensions, including those of spacetime. This constellation of observations suggests the viewpoint of spacetime as a kind of dislocation continuum with its 4-d Cartan torsion tensor related algebraically to the intrinsic spin angular momentum of a material medium.

For more on the line of reasoning leading to the above observations and conclusions, see "Spin and the Structure of Spacetime" by F. Hehl and P. von der Heyde in Ann. Inst. Henri Poincare, Vol. XIX, n°2, 1973, p179-196.