

LECTURE 35

35.1

I. Cartan's 2nd Structure Equation

II. Cartan's Unit-economical Derivation of his Two Structure Equations.

In MTW read P 348-354 (§ 14.5)

Comment:

A comparison between MTW's Eq. (14.39) and (14.20) will lead to confusion because their "R" in Eq. (14.39) is a matrix of 2-forms. By contrast, their "R" in Eq. (14.20), (15.12), (15.14), (15.16), (15.19)-(15.21), (15.25), (15.26) is a dyad (i.e. a $\binom{1}{2}$ or $\binom{2}{2}$ tensor)-valued 2-form. To avoid this confusion and to highlight the dyad-valued nature of "R" in these equations, MTW should have replaced that "R" with \vec{R} instead.

I. Cartan's 2nd Structure Equation

The Riemann curvature tensor is the point-wise linear map,

$$R : T_p(M) \times T_p(M) \times T_p(M) \rightarrow T_p(M),$$

$$(W, U, V) \rightsquigarrow R(U, W, U, V) = \{ \nabla_U \nabla_V - \nabla_V \nabla_U - [U, V] \} W \equiv \vec{R}(U, V) W$$

which produces an amount of curvature-induced vectorial rotation from a given vector that has been parallel transported around the perimeter of the area enclosed by the curve segments of two vector fields u, v , and their commutator $[u, v]$,

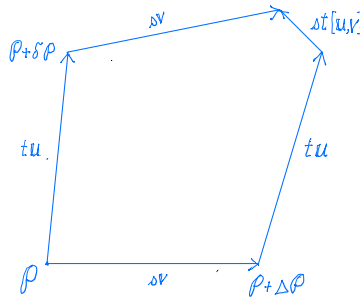


Figure 34.1: A closed curve as determined by two vector fields and their commutator

Arrive at Cartan's 2nd structure equation by using the same kind of algebraic and calculus reasoning as in his 1st equation. Do this by using the modern post-WWII approach, i.e. Cartan's formulation of Stokes' theorem in terms of his exterior derivative applied to any differential 1-form ω , namely

$$u \langle \omega | v \rangle - v \langle \omega | u \rangle - \langle \omega | [u, v] \rangle = d\omega(u, v).$$

Then obtain the structure equation in four steps:

STEP 1.

Focus separately on each of the three terms of the curvature-induced rotation mapping,

$$\left\{ \begin{array}{l} \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]} \end{array} \right\} W = \textcircled{1} - \textcircled{2} - \textcircled{3},$$

where

$$\left. \begin{array}{l} \textcircled{1} = \nabla_u \nabla_v (e_k w^k) \\ \textcircled{2} = \nabla_v \nabla_u (e_k w^k) \\ \textcircled{3} = \nabla_{[u,v]} (e_k w^k) \end{array} \right\} w^k = \langle \underline{\omega}^k | W \rangle \text{ with } \langle \underline{\omega}^k | e_l \rangle = \delta^k_l$$

STEP 2.

Using the Leibnitz product rule of the covariant derivative ∇ , find that all 1st and 2nd derivative terms referring to $u(w^k)$, $v(w^k)$, $uv(w^k)$, and $vu(w^k)$ cancel.

STEP 3.

A) Introduce the Cartan connection 1-forms $\underline{\omega}^j_k$,

$$\nabla_u (e_k) = e_l \langle \underline{\omega}^l_k | u \rangle$$

$$\nabla_v (e_k) = e_l \langle \underline{\omega}^l_k | v \rangle$$

into the remaining terms of $\textcircled{1} - \textcircled{2} - \textcircled{3}$. The result is

$$\langle \underline{\omega}^l_j \otimes \underline{\omega}^j_k - \underline{\omega}^j_k \otimes \underline{\omega}^l_j \rangle (u, v) = \omega^l_j \wedge \omega^j_k (u, v)$$

$$\begin{aligned} \textcircled{1} - \textcircled{2} - \textcircled{3} &= e_l \left[\langle \underline{\omega}^l_j | u \rangle \langle \underline{\omega}^j_k | v \rangle - \langle \underline{\omega}^l_j | v \rangle \langle \underline{\omega}^j_k | u \rangle \right] w^k \\ &\quad + e_l \left[u \langle \underline{\omega}^l_k | v \rangle - v \langle \underline{\omega}^l_k | u \rangle - \langle \underline{\omega}^l_k | [u, v] \rangle \right] w^k \end{aligned}$$

$$d\omega^l_R(u, v)$$

The first two terms consolidate into the wedge product $e_l \omega^k \omega^j \wedge \omega^l_R$ evaluated on (u, v) .

Referring to Eq. (33.4) on page 33.6 of Lecture 33, note that the last three terms are proportional to $d\omega^l_R$, the exterior derivative of ω^l_R evaluated on (u, v) .

Thus, in ① - ② - ③ all 1st derivative terms referring to $u(v^k)$ and $v(u^k)$ cancel.

B.) The final (point-wise linear) result is

$$\textcircled{1} - \textcircled{2} - \textcircled{3} = e_l \omega^k [d\omega^l_R + \omega^j \wedge \omega^j_R](u, v)$$

or, in light of $\omega^k = \omega^k(w)$,

$$\textcircled{1} - \textcircled{2} - \textcircled{3} = e_l \otimes \omega^k [d\omega^l_R + \omega^j \wedge \omega^j_R](w, u, v)$$

$$\equiv \mathbb{R}(\dots; w, u, v)$$

Here $\mathbb{R} = e_l \otimes \omega^k \otimes [d\omega^l_R + \omega^j \wedge \omega^j_R]$

the Riemann tensor. It is simply a tensor of rank (3).

Being a multilinear map, the Riemann tensor \mathbb{R} has well-determined components relative a chosen basis. They are

$$\mathbb{R}(\omega^r, e_s, e_m, e_n) \equiv R^r_{smn}$$

$$= \langle \omega^r | [\nabla_{e_m} \nabla_{e_n} - \nabla_{e_n} \nabla_{e_m} - \nabla_{[e_m, e_n]}] e_s \rangle$$

$$= (d\omega^r_s + \omega^r_j \wedge \omega^j_s)(e_m, e_n)$$

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STEP 4.

However, Cartan initiated a more penetrating and fruitful perspective on this multilinear map. His view is to rewrite this tensor in the form

$$R = e_\ell \otimes \omega^k \underline{R}^{\ell}_{\ k}$$

where

$$\underline{R}^{\ell}_{\ k} \equiv d\omega^{\ell}_{\ k} + \omega^{\ell}_j \wedge \omega^j_k$$

and collectively call this the curvature 2-form.

According to this perspective,

$$R = \vec{R} = e_\ell \otimes \omega^k \underline{R}^{\ell}_{\ k}$$

is a tensor-valued two-form. When it is evaluated on a pair of area spanning vectors u and v , the result is the tensor

$$\vec{R}(u, v) = e_\ell \otimes \omega^k \underline{R}^{\ell}_{\ k}(u, v).$$

It is the generator of rotations of vectors parallel transported around the perimeter of the (curvature permeated) area spanned by u and v . If w is one of those vectors, then it will undergo the vectorial change

$$\vec{R}(u, v)(w) = e_\ell \omega^k \underline{R}^{\ell}_{\ k}(u, v) w^k.$$

The infinitesimally rotated vector is

$$[\vec{I} + \vec{R}(u, v)]w = e_\ell [\delta^{\ell}_{\ k} + \underline{R}^{\ell}_{\ k}(u, v)] w^k.$$

CONCLUSION

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The tensor-valued curvature two-form

$$\begin{aligned} \mathbb{R} = \vec{R} &= e_R \otimes \omega^l (d\omega^k + \omega^k_j \wedge \omega^j) = e_R \otimes \omega^l R^k_{l[mn]} \omega^m \wedge \omega^n \\ &= e_R \otimes \omega^l \frac{1}{2} R^k_{l\ mn} \omega^m \wedge \omega^n \end{aligned}$$

is Cartan's second structure equation.

This first structure equation is the vector-valued torsion two-form

$$\begin{aligned} \mathbb{T} = \vec{T} &= e_R (d\omega^k + \omega^k_j \wedge \omega^j) = e_R T^k_{[mn]} \omega^m \wedge \omega^n \\ &= e_R \frac{1}{2} T^k_{mn} \omega^m \wedge \omega^n \end{aligned}$$

II. The 1st and 2nd structure equation via Cartan's line of reasoning (35.7)
 He obtained his two equations in a unit-economical way by applying the exterior derivative to

1.) his unit tensor

$$e_i \otimes \omega^i = d\rho \quad (35.1)$$

and

2.) his mathematization parallel transport

$$de_i = e_j \otimes \omega^j_i \quad (35.2)$$

He obtains the 1st structure equation as follows:

Take the exterior derivative of Eq. (35.1) and use Eq. (35.2):

$$\begin{aligned} d(d\rho) &= d(e_i \otimes \omega^i) \\ &= e_i \otimes d\omega^i + de_i \wedge \omega^i \end{aligned}$$

$$d^2\rho = e_j \otimes (d\omega^j + \omega^j_i \wedge \omega^i) = e_j T^j_{|mn|} \omega^m \wedge \omega^n$$

He obtains the 2nd structure equation as follows:

Take the exterior derivative of Eq. (35.2) and use Eq. (35.2):

$$\begin{aligned} d(de_i) &= d(e_j \otimes \omega^j_i) \\ &= e_j d\omega^j_i + de_j \wedge \omega^j_i \end{aligned}$$

$$d^2e_i = e_R \otimes (d\omega^R_i + \omega^R_j \wedge \omega^j_i) = e_R R^R_{i|mn|} \omega^m \wedge \omega^n$$

Question: What is the essence of the cause of his successful line of reasoning?

Answer: ...