LECTURE 38

I. The Cartan-Misner Calculus

1. From metric to curvature

2. Physical (a.k.a. orthonormal) frames.

II. Coordinate vs. Orthonormal Basis Expansions

I. The 4 Fundamental Equations of Differential Geometry: Curvature via the Cartan-Misner Calculus.

The modern way of mathematizing the key geometrical objects of differential geometry is based on Cartan's formulation (via his exterior calculus) of the 4 structure equations of differential geometry, i.e.

GIVEN:

$$dP = e_i\,\omega^i;\quad de_i = e_j\,\omega^j{}_i;\quad e_i\cdot e_j = g_{ij}$$

the 4 fundamental structure equations are

① $d\omega^i + \omega^i{}_j\wedge\omega^j \equiv \Omega^i = T^i{}_{|mn|}\,\omega^m\wedge\omega^n$

② $d\omega^i{}_j + \omega^i{}_k\wedge\omega^k{}_j \equiv \Omega^i{}_j = R^i{}_{j|mn|}\,\omega^m\wedge\omega^n$

③ $d(e_i\cdot e_j) \equiv dg_{ij} = \omega_{ij} + \omega_{ji}$, where $\omega_{ij} = g_{ik}\,\omega^k{}_j$

④ $0 = d(dg_{ij}) = \Omega_{ij} + \Omega_{ji}$, where $\Omega_{ij} = g_{ik}\,\omega^k{}_j$

The process of calculating all components of connection 1-forms $\omega^i{}_j$, and hence of the curvature tensor using the Cartan-Misner calculus is a three step process:

STEP 1.

Given a metric construct an orthonormal basis.

If $[g_{\mu\nu}]$ is diagonal:

$$g = g_{\mu\nu}\,dx^\mu\otimes dx^\nu = \hat{g}_{\mu\nu}\,\hat{\omega}^\mu\otimes\hat{\omega}^\nu$$
$$= g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2$$
$$= -(\hat{\omega}^0)^2 + (\hat{\omega}^1)^2 + (\hat{\omega}^0)^2 + (\hat{\omega}^3),$$

let

$$[\hat{g}_{\mu\nu}] = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \quad \text{and} \quad \hat{\omega}^\mu = |g_{\mu\mu}|^{1/2} dx^\mu, \qquad \mu = 0,1,2,3$$

STEP 2.

Solve for $\hat{\omega}^\mu{}_\nu$ from

a) the zero torsion condition

$$d\hat{\omega}^\mu + \hat{\omega}^\mu{}_\nu \wedge \hat{\omega}^\nu = 0 \longrightarrow \boxed{\hat{\omega}^\nu \wedge \omega^\mu{}_\nu = d\omega^\mu}$$

and b) the metric compatibility condition

$$d\hat{g}_{\mu\nu} = 0 = \hat{\omega}_{\mu\nu} + \hat{\omega}_{\nu\mu} \longrightarrow \boxed{\hat{\omega}_{\mu\nu} = -\hat{\omega}_{\nu\mu}}$$

STEP 3.

Calculate the curvature 2-forms and read out the curvature components for each of them:

$$\hat{\Omega}^\mu{}_\nu \equiv d\hat{\omega}^\mu{}_\nu + \hat{\omega}^\mu{}_\gamma \wedge \hat{\omega}^\gamma{}_\nu = \hat{R}^\mu{}_{\nu|\sigma\rho|} \hat{\omega}^\sigma \wedge \hat{\omega}^\rho$$

As a computational check to verify that

$$\hat{\Omega}_{\mu\nu} = -\hat{\Omega}_{\nu\mu} \longrightarrow \hat{R}_{\mu\nu\sigma\rho} = -\hat{R}_{\nu\mu\sigma\rho}$$

Remark.

Einstein's 1916 geometrization of gravitation consists of having the spacetime distribution of momentum and energy determine the curvature properties (more precisely, the curvature-induced "moment of rotation") of spacetime. Mathematizing geometry is a matter of deriving curvature from the metric tensor field (i.e. the geometrized Newtonian gravitational potential).

Consequently, during the two or three decades after 1916 there was (and still is) great interest in calculating the curvature from a to-be-determined metric tensor field. By necessity these calculations took the form

metric ⟶ Christoffel symbols ⟶ curvature

As illustrated by Dingle's 1933 article (copied below), the calculation of the Christoffel symbols and of their derivatives in terms of the first and second derivatives of the metric coefficients is not only a daunting task, but the proliferation of derivatives detracts from understanding.

Dingle satisfies metric compatibility of parallel transport by computing the Christoffel symbols in retail,

$$\Gamma^i{}_{jk} = \frac{g^{il}}{2}\left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right),$$

one $\Gamma^i{}_{jk}$ at a time.

Cartan and Misner, by contrast, satisfy compatibility wholesale. Misner (see the Appendix of his paper copied below) states metric compatibility in the form of unique solutions to the systems of linear equations

$$\omega^j \wedge \omega^i{}_k = d\omega^i$$

$$g_{il}\,\omega^l{}_j + g_{jl}\,\omega^l{}_i = dg_{ij}$$

whose unknowns $\omega^j{}_i$ are to be determined. Compared to Dingle's

retail approach, this is a task easier by an order (38,5)
order of magnitude in 4-d spacetime manifolds.

VALUES OF T^ν_μ AND THE CHRISTOFFEL SYMBOLS FOR A LINE ELEMENT OF CONSIDERABLE GENERALITY

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In the general theory of relativity the mechanical properties of any region of the universe are expressed by the energy-momentum tensor, T^ν_μ, which is itself calculable from the form of the line element, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, applicable to that region. The expressions for T^ν_μ in the most general case, in which all the $g_{\mu\nu}$ are arbitrary functions of the four coördinates, x^1, x^2, x^3, x^4, are exceedingly complicated, but considerable simplification is introduced if it is assumed that $g_{\mu\nu}(\mu \neq \nu) = 0$. The resulting line element still possesses a large amount of generality, and in the applications of the theory particular forms of it have, in fact, usually been employed. It therefore seems desirable to publish the general expressions for the energy-momentum tensor corresponding to this line element, and it is the purpose of this paper to give them, together with the associated values of the Christoffel symbols of the second kind, in the form best suited for application. The calculations, which are somewhat long, have kindly been checked by Mr. C. C. Steffens of the California Institute of Technology, and the proofs have been carefully read, so that the results may be used with considerable confidence. It is hoped that their publication will save labor for those working in this field.

The expression for the line element is taken as

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2,$$

where A, B, C and D are any functions of x^1, x^2, x^3 and x^4. Mathematically these functions may be positive or negative, real or imaginary,* but in ordinary applications, in which x^4 is the time-like coördinate, they will clearly always be positive and real. The non-vanishing components of the metric tensor and its contravariant associate are obviously as follows:

$$g_{11} = -A;\ g_{22} = -B;\ g_{33} = -C;\ g_{44} = +D$$

$$g^{11} = -\frac{1}{A};\ g^{22} = -\frac{1}{B};\ g^{33} = -\frac{1}{C};\ g^{44} = +\frac{1}{D};$$

and the determinant, g, is $-ABCD$.

Christoffel Symbols.—These are defined by the expression

$$\{\mu\nu, \sigma\} = \tfrac{1}{2}\, g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right).$$

* It is assumed that they possess first and second differential coefficients with respect to each of the coördinates.

Their values are

$\{11,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^1}$	$\{21,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^2}$	$\{31,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^3}$	$\{41,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^4}$
$\{11,2\} = -\frac{1}{2B}\frac{\partial A}{\partial x^2}$	$\{21,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^1}$	$\{31,2\} = 0$	$\{41,2\} = 0$
$\{11,3\} = -\frac{1}{2C}\frac{\partial A}{\partial x^3}$	$\{21,3\} = 0$	$\{31,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^1}$	$\{41,3\} = 0$
$\{11,4\} = +\frac{1}{2D}\frac{\partial A}{\partial x^4}$	$\{21,4\} = 0$	$\{31,4\} = 0$	$\{41,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^1}$
$\{12,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^2}$	$\{22,1\} = -\frac{1}{2A}\frac{\partial B}{\partial x^1}$	$\{32,1\} = 0$	$\{42,1\} = 0$
$\{12,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^1}$	$\{22,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^2}$	$\{32,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^3}$	$\{42,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^4}$
$\{12,3\} = 0$	$\{22,3\} = -\frac{1}{2C}\frac{\partial B}{\partial x^3}$	$\{32,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^2}$	$\{42,3\} = 0$
$\{12,4\} = 0$	$\{22,4\} = +\frac{1}{2D}\frac{\partial B}{\partial x^4}$	$\{32,4\} = 0$	$\{42,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^2}$
$\{13,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^3}$	$\{23,1\} = 0$	$\{33,1\} = -\frac{1}{2A}\frac{\partial C}{\partial x^1}$	$\{43,1\} = 0$
$\{13,2\} = 0$	$\{23,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^3}$	$\{33,2\} = -\frac{1}{2B}\frac{\partial C}{\partial x^2}$	$\{43,2\} = 0$
$\{13,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^1}$	$\{23,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^2}$	$\{33,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^3}$	$\{43,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^4}$
$\{13,4\} = 0$	$\{23,4\} = 0$	$\{33,4\} = +\frac{1}{2D}\frac{\partial C}{\partial x^4}$	$\{43,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^3}$
$\{14,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^4}$	$\{24,1\} = 0$	$\{34,1\} = 0$	$\{44,1\} = +\frac{1}{2A}\frac{\partial D}{\partial x^1}$
$\{14,2\} = 0$	$\{24,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^4}$	$\{34,2\} = 0$	$\{44,2\} = +\frac{1}{2B}\frac{\partial D}{\partial x^2}$
$\{14,3\} = 0$	$\{24,3\} = 0$	$\{34,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^4}$	$\{44,3\} = +\frac{1}{2C}\frac{\partial D}{\partial x^3}$
$\{14,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^1}$	$\{24,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^2}$	$\{34,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^3}$	$\{44,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^4}$

Energy-Momentum Tensor, T^ν_μ.—This tensor is defined by the expression

$$-8\pi T^\nu_\mu = G^\nu_\mu - \tfrac{1}{2}\, g^\nu_\mu G + g^\nu_\mu \lambda$$

where G^ν_μ is the contracted Riemann-Christoffel tensor, G is the invariant,

$g^{\mu\nu}G_{\mu\nu}$, and λ is the cosmological constant. The values of $-8\pi T^{\nu}_{\mu}$ are as follows:

$$-8\pi T^1_1 = \frac{1}{2}\left[\frac{1}{BC}\left(\frac{\partial^2 B}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^2)^2}\right)-\frac{1}{BD}\left(\frac{\partial^2 B}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^2)^2}\right)-\frac{1}{CD}\left(\frac{\partial^2 C}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^3)^2}\right)\right]$$

$$-\frac{1}{4}\left[\frac{1}{BC^2}\left\{\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^2}\right)^2\right\}+\frac{1}{CB^2}\left\{\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^3}\right)^2\right\}\right.$$
$$-\frac{1}{BD^2}\left\{\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^2}\right)^2\right\}+\frac{1}{DB^2}\left\{\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^2}-\left(\frac{\partial B}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{CD^2}\left\{\frac{\partial C}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^3}\right)^2\right\}+\frac{1}{DC^2}\left\{\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^3}-\left(\frac{\partial C}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{BCD}\left\{\frac{\partial C}{\partial x^2}\frac{\partial D}{\partial x^2}+\frac{\partial B}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{\partial B}{\partial x^4}\frac{\partial C}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1}$$
$$\left.-\frac{1}{ABD}\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^1}-\frac{1}{ACD}\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^1}\right]+$$

$$-8\pi T^2_2 = \frac{1}{2}\left[\frac{1}{AC}\left(\frac{\partial^2 A}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^1)^2}\right)-\frac{1}{AD}\left(\frac{\partial^2 A}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^1)^2}\right)-\frac{1}{CD}\left(\frac{\partial^2 C}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^3)^2}\right)\right]$$

$$-\frac{1}{4}\left[\frac{1}{AC^2}\left\{\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^1}\right)^2\right\}+\frac{1}{CA^2}\left\{\frac{\partial C}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^3}\right)^2\right\}\right.$$
$$-\frac{1}{AD^2}\left\{\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^1}\right)^2\right\}+\frac{1}{DA^2}\left\{\frac{\partial D}{\partial x^1}\frac{\partial A}{\partial x^1}-\left(\frac{\partial A}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{CD^2}\left\{\frac{\partial C}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^3}\right)^2\right\}+\frac{1}{DC^2}\left\{\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^3}-\left(\frac{\partial C}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{ACD}\left\{\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^1}+\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2}$$
$$\left.-\frac{1}{ABD}\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^2}-\frac{1}{BCD}\frac{\partial C}{\partial x^2}\frac{\partial D}{\partial x^2}\right]+\lambda$$

$$-8\pi T^3_3 = \frac{1}{2}\left[\frac{1}{AB}\left(\frac{\partial^2 A}{\partial(x^2)^2}+\frac{\partial^2 B}{\partial(x^1)^2}\right)-\frac{1}{AD}\left(\frac{\partial^2 A}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^1)^2}\right)-\frac{1}{BD}\left(\frac{\partial^2 B}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^2)^2}\right)\right]$$

$$-\frac{1}{4}\left[\frac{1}{AB^2}\left\{\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^1}\right)^2\right\}+\frac{1}{BA^2}\left\{\frac{\partial B}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^2}\right)^2\right\}\right.$$

$$-\frac{1}{AD^2}\left\{\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^1}\right)^2\right\}+\frac{1}{DA^2}\left\{\frac{\partial D}{\partial x^1}\frac{\partial A}{\partial x^1}-\left(\frac{\partial A}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{BD^2}\left\{\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^2}\right)^2\right\}+\frac{1}{DB^2}\left\{\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^2}-\left(\frac{\partial B}{\partial x^4}\right)^2\right\}$$
$$-\frac{1}{ABD}\left\{\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^1}+\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^2}-\frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}$$
$$\left.-\frac{1}{ACD}\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{1}{BCD}\frac{\partial B}{\partial x^3}\frac{\partial D}{\partial x^3}\right]+\lambda$$

$$-8\pi T_4^4=\frac{1}{2}\left[\frac{1}{AB}\left(\frac{\partial^2 A}{\partial(x^2)^2}+\frac{\partial^2 B}{\partial(x^1)^2}\right)+\frac{1}{AC}\left(\frac{\partial^2 A}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^1)^2}\right)\right.$$
$$\left.+\frac{1}{BC}\left(\frac{\partial^2 B}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^2)^2}\right)\right]$$
$$-\frac{1}{4}\left[\frac{1}{AB^2}\left\{\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^1}\right)^2\right\}+\frac{1}{BA^2}\left\{\frac{\partial B}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^2}\right)^2\right\}\right.$$
$$+\frac{1}{AC^2}\left\{\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^1}\right)^2\right\}+\frac{1}{CA^2}\left\{\frac{\partial C}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^3}\right)^2\right\}$$
$$+\frac{1}{BC^2}\left\{\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^2}\right)^2\right\}+\frac{1}{CB^2}\left\{\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^3}\right)^2\right\}$$
$$-\frac{1}{ABC}\left\{\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1}+\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2}+\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right\}+\frac{1}{ABD}\frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^4}$$
$$\left.+\frac{1}{ACD}\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^4}+\frac{1}{BCD}\frac{\partial B}{\partial x^4}\frac{\partial C}{\partial x^4}\right]+\lambda$$

$$-8\pi AT_2^1=-8\pi BT_1^2=$$
$$-\frac{1}{2}\left[\frac{1}{C}\frac{\partial^2 C}{\partial x^1\partial x^2}+\frac{1}{D}\frac{\partial^2 D}{\partial x^1\partial x^2}\right]$$
$$+\frac{1}{4}\left[\frac{1}{C^2}\frac{\partial C}{\partial x^1}\frac{\partial C}{\partial x^2}+\frac{1}{D^2}\frac{\partial D}{\partial x^1}\frac{\partial D}{\partial x^2}+\frac{1}{AC}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^1}+\frac{1}{AD}\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^1}\right.$$
$$\left.+\frac{1}{BC}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^2}+\frac{1}{BD}\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^2}\right]$$

$$-8\pi AT_3^1=-8\pi CT_1^3=$$
$$-\frac{1}{2}\left[\frac{1}{B}\frac{\partial^2 B}{\partial x^1\partial x^3}+\frac{1}{D}\frac{\partial^2 D}{\partial x^1\partial x^3}\right]$$
$$+\frac{1}{4}\left[\frac{1}{B^2}\frac{\partial B}{\partial x^1}\frac{\partial B}{\partial x^3}+\frac{1}{D^2}\frac{\partial D}{\partial x^1}\frac{\partial D}{\partial x^3}+\frac{1}{AB}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^1}+\frac{1}{AD}\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^1}\right.$$
$$\left.+\frac{1}{CB}\frac{\partial C}{\partial x^1}\frac{\partial B}{\partial x^3}+\frac{1}{CD}\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^3}\right]$$

$$-8\pi BT_3^2 = -8\pi CT_2^3 =$$
$$-\frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2 \partial x^3} + \frac{1}{D}\frac{\partial^2 D}{\partial x^2 \partial x^3}\right]$$
$$+\frac{1}{4}\left[\frac{1}{A^2}\frac{\partial A}{\partial x^2}\frac{\partial A}{\partial x^3} + \frac{1}{D^2}\frac{\partial D}{\partial x^2}\frac{\partial D}{\partial x^3} + \frac{1}{AB}\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^3} + \frac{1}{AC}\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^2}\right.$$
$$\left. + \frac{1}{DB}\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^3} + \frac{1}{DC}\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^2}\right]$$

$$-8\pi AT_4^1 = +8\pi DT_1^4 =$$
$$-\frac{1}{2}\left[\frac{1}{B}\frac{\partial^2 B}{\partial x^1 \partial x^4} + \frac{1}{C}\frac{\partial^2 C}{\partial x^1 \partial x^4}\right]$$
$$+\frac{1}{4}\left[\frac{1}{B^2}\frac{\partial B}{\partial x^1}\frac{\partial B}{\partial x^4} + \frac{1}{C^2}\frac{\partial C}{\partial x^1}\frac{\partial C}{\partial x^4} + \frac{1}{AB}\frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^1} + \frac{1}{AC}\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^1}\right.$$
$$\left. + \frac{1}{DB}\frac{\partial D}{\partial x^1}\frac{\partial B}{\partial x^4} + \frac{1}{DC}\frac{\partial D}{\partial x^1}\frac{\partial C}{\partial x^4}\right]$$

$$-8\pi BT_4^2 = +8\pi DT_2^4 =$$
$$-\frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^2 \partial x^4} + \frac{1}{C}\frac{\partial^2 C}{\partial x^2 \partial x^4}\right]$$
$$+\frac{1}{4}\left[\frac{1}{A^2}\frac{\partial A}{\partial x^2}\frac{\partial A}{\partial x^4} + \frac{1}{C^2}\frac{\partial C}{\partial x^2}\frac{\partial C}{\partial x^4} + \frac{1}{AB}\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^4} + \frac{1}{AD}\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^2}\right.$$
$$\left. + \frac{1}{CB}\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^4} + \frac{1}{DC}\frac{\partial D}{\partial x^2}\frac{\partial C}{\partial x^4}\right]$$

$$-8\pi CT_4^3 = +8\pi DT_3^4 =$$
$$-\frac{1}{2}\left[\frac{1}{A}\frac{\partial^2 A}{\partial x^3 \partial x^4} + \frac{1}{B}\frac{\partial^2 B}{\partial x^3 \partial x^4}\right]$$
$$+\frac{1}{4}\left[\frac{1}{A^2}\frac{\partial A}{\partial x^3}\frac{\partial A}{\partial x^4} + \frac{1}{B^2}\frac{\partial B}{\partial x^3}\frac{\partial B}{\partial x^4} + \frac{1}{AC}\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^4} + \frac{1}{AD}\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^3}\right.$$
$$\left. + \frac{1}{BC}\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^4} + \frac{1}{BD}\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^3}\right]$$

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The Flatter Regions of Newman, Unti, and Tamburino's Generalized Schwarzschild Space*

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The "generalized Schwarzschild" metric discovered by Newman, Unti, and Tamburino, which is stationary and spherically symmetric, is investigated. We find that the orbit of a point under the group of time translations is a circle, rather than a line as in the Schwarzschild case. The timelike hypersurfaces r = const which are left invariant by the group of motions are topologically three-spheres S^3, in contrast to the topology $S^2 \times R$ (or $S^2 \times S^1$) for the r = const surfaces in the Schwarzschild case. In the Schwarzschild case, the intersection of a spacelike surface t = const and an r = const surface is a sphere S^2. If σ is any spacelike hypersurface in the generalized metric, then its (two-dimensional) intersection with an r = const surface is not any closed two-dimensional manifold, that is, the generalized metric admits no reasonable spacelike surfaces. Thus, even though all curvature invariants vanish as $r \to \infty$, in fact $R_{\mu\nu\alpha\beta} = O(1/r^3)$ as in the Schwarzschild case, this metric is not asymptotically flat in the sense that coordinates can be introduced for which $g_{\mu\nu} - \eta_{\mu\nu} = O(1/r)$. An apparent singularity in the metric at small values of r, which appears to be similar to the spurious Schwarzschild singularity at $r = 2m$, has not been studied. If this singularity should again be spurious, then the "generalized Schwarzschild" space would represent a terminal phase in the evolution of an entirely nonsingular cosmological model which, in other phases, contains closed spacelike hypersurfaces but no matter.

I. INTRODUCTION

THE primary purpose of this paper is to study and describe geometrically the stationary, spherically symmetric solution of the Einstein equations recently discovered by Newman, Unti, and Tamburino[1] which I shall refer to as NUT space. A second important purpose of this work is to provide an example of the recognition and elimination of a spurious singularity in a Riemannian line element with the Lorentz $-+++$ signature. No general method is known for eliminating coordinate singularities in a metric, nor are there adequate criteria to determine that a singularity is not merely a coordinate singularity, and I expect that further examples beyond the Kruskal–Fronsdal[2,3] elimination of the Schwarzschild singularity will be helpful in leading to an understanding of these problems. A third, minor, aim of this paper is to provide an example of the use of orthonormal frames (tetrads) in a style more economical with indices than is usual in the literature of physics, and in particular a method of computing the curvature tensor very rapidly (cf. Appendix A).

The question of singularities in metrics is broader and more important than the study of one particular metric to which most of this paper is devoted, so I will briefly summarize the present state of the art.[4] The first step is to find some clearly stated problems, and the clue to clarity is to refuse ever to speak of a singularity but instead to phrase everything in terms of the properties of differentiable metric fields on manifolds. If one is given a manifold, and on it a metric which does not at all points satisfy the necessary differentiability requirements, one simply throws away all the points of singularity. The starting point for any further discussion is then the largest submanifold on which the metric *is* differentiable. This is done because there is not known any useful way of describing the singularities of a function except by describing its behavior at regular points near the singularity. The first problem then is to select a criteria which will identify in an intuitively acceptable way a "nonsingular space." Evidently, differentiability is only a minimum prerequisite, since everything becomes differentiable when the singular points are discarded. The problem is rather to recognize the holes left in the space where singular (or even regular) points have been omitted. For a connected Riemannian manifold

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[1] E. Newman, L. Tamburino, and T. Unti, J. Math. Phys. 4, 915 (1963). I wish to thank these authors for sending me a preprint of their paper.

[2] M. D. Kruskal, Phys. Rev. 119, 1743 (1960).

[3] C. Fronsdal, Phys. Rev. 116, 778 (1959).

[4] I wish to thank Mr. L. Shepley for preparing this review and for correcting numerous errors in an earlier draft. We have borrowed heavily from L. Marcus' lectures on this topic at the American Mathematical Society's 1962 Summer Institute at the University of California at Santa Barbara.

free space–time. As yet, no example is known of a nonflat singularity-free cosmological solution with vanishing cosmological constant. The behavior we are led to conjecture for an extended NUT space would also be quite remarkable. The unequal expansion rates in different directions of the closed spacelike hypersurface $r = 0$ would smoothly develop into a situation where closed spacelike hypersurfaces no longer could exist, while the evolution in time would smoothly resolve itself into a state of affairs which was periodic in time. A difficulty which will arise in attempting to eliminate the Schwarzschild-like singularities at $f^2 = 0$ using Kruskal's methods is the time periodicity of NUT space. For instance, in the Schwarzschild solution for $r > 2m$ (or for $0 < r < 2m$), we can easily identify points to give a periodic time, $t \equiv t + T$. In Kruskal's extended Schwarzschild solution,[2] we may then attempt to make this same identification. The singularity in the coordinate t causes no difficulty, since it is not the t coordinate, but motions along the Killing vector field $\partial/\partial t$ which define the identifications we wish to make, and $\partial/\partial t$ is an analytic vector field, everywhere free from singularities, as is evident from its expression in terms of Kruskal's nonsingular coordinates u, v:

$$\frac{\partial}{\partial t} = \frac{1}{4m}\left(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right). \tag{57}$$

What happens, then, when we identify points which differ by a motion of amount T, i.e. when we identify points P and $\exp\{T\,\partial/\partial t\}P$? In fact, this introduces a singularity, not everywhere along the null 3-surfaces $r = 2m$ (or $u = \pm v$), but only at the single 2-surface $u = v = 0$ where, by Eq. (57), $\partial/\partial t = 0$ so the Killing motion has fixed points.

APPENDIX A. COMPUTATION OF THE CURVATURE

We shall compute the Riemann tensor by methods due to Cartan[20] which, at least for metrics with a considerable amount of symmetry, are much more efficient than the methods usually employed by physicists. If $\omega^1, \omega^2, \cdots \omega^n$ are a set of covariant basis vectors, then the metric tensor is written

$$ds^2 = g_{\mu\nu}\omega^\mu\omega^\nu. \tag{A1}$$

Covariant derivatives are formed with the aid of the connection forms $\omega^\mu{}_\nu$, or their components $\Gamma^\mu{}_{\alpha\beta}$:

$$\omega^\mu{}_\alpha = \Gamma^\mu{}_{\alpha\beta}\,\omega^\beta. \tag{A2}$$

[20] E. Cartan [cf. reference 10, Chap. VII]; T. J. Willmore [cf. reference 5, Secs. VII-16 and VII-19]; H. Flanders, Trans. Am. Math. Soc. **75**, 311 (1953).

The following two sets of equations determine the $\omega^\mu{}_\alpha$ uniquely:

$$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu} \tag{A3}$$

$$d\omega^\mu = -\omega^\mu_\nu \wedge \omega^\nu. \tag{A4}$$

In the more familiar case of a *coordinate* frame $\omega^\mu = dx^\mu$, the second equation here gives $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$ (using the property $d^2 = 0$ of the exterior derivative, and the antisymmetry of the exterior product $\wedge$), while the first is a standard relation between the metric derivatives and the $\Gamma_{\mu\alpha b}$, which is solved to show that $\Gamma^\mu{}_{\alpha\beta}$ is a Christoffel symbol. We will use these equations in a different case, that of an orthonormal frame, where, since $g_{\mu\nu} = \eta_{\mu\nu} = \text{const}$, Eq. (A3) states that the forms $\omega_{\mu\nu}$ are anti-symmetric

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0. \tag{A5}$$

With the aid of this antisymmetry, Eqs. (A4) can now be solved for $\omega^\mu{}_\nu$ when ω^μ is given. Although a formula like the Christoffel relation exists also in this case, *the computation is most efficient when Eqs. (A) can be solved by inspection*, as we shall shortly illustrate. Once the connection forms have been computed, the curvature forms $\theta^\mu{}_\nu$ are obtained from the formula

$$\theta^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu. \tag{A6}$$

The components of the Riemann tensor $R^\mu{}_{\nu\alpha\beta}$ are then read out of these curvature froms:

$$\theta^\mu{}_\nu = \tfrac{1}{2}R^\mu{}_{\nu\alpha\beta}\,\omega^\alpha \wedge \omega^\beta, \tag{A7}$$

and the Ricci tensor is formed by contraction

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \tag{A8}$$

Note that in 4-space, with an *orthonormal* basis ω^μ, there are only six connection forms $\omega^\mu{}_\nu$ in contrast to forty Christoffel symbols, and only six curvature forms $\theta^\mu{}_\nu$ in contrast to twenty components $R_{\mu\nu\alpha b}$ of the Riemann tensor or ten for the Ricci tensor. In simple cases, such as NUT space, a savings of labor on a scale suggested by these numbers is actually attained, and the Ricci tensor can be computed much more rapidly by these methods (which provide the Riemann tensor as a bonus along the way) than from the usual formula in terms of Christoffel symbols.

The computation begins by writing the metric in terms of an orthonormal frame, as has been done in Eqs. (7) and (8). Next the curl, $d\omega^\mu$, of each base vector ω^μ must be computed. Let us compute, for example, $d\omega^0$ where

$$\omega^0 = f(r)[dt + 4l\sin^2\tfrac{1}{2}\theta\, d\phi]. \tag{A9}$$

First note that $d(f[\ ]) = df \wedge [\ ] + fd[\ ]$, and $df = f'dr = f'f\omega^1$ since $\omega^1 = f^{-1}dr$. Thus $d\omega^0 = f'\omega^1 \wedge \omega^0 + fd[\ ]$. Using $d^2 = 0$ we then find

$$d[\ ] = 4l \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta \, d\theta \wedge d\phi = 2l \sin\theta \, d\theta \wedge d\phi = 2l(r^2 + l^2)^{-1}\omega^2 \wedge \omega^3.$$

Proceeding in this manner we obtain

$$\begin{aligned} d\omega^0 &= f'\omega^1 \wedge \omega^0 + 2lf(r^2 + l^2)^{-1}\omega^2 \wedge \omega^3, \\ d\omega^1 &= 0, \\ d\omega^2 &= rf(r^2 + l^2)^{-1}\omega^1 \wedge \omega^2, \\ d\omega^3 &= rf(r^2 + l^2)^{-1}\omega^1 \wedge \omega^3 + (r^2 + l^2)^{-\frac{1}{2}} \cot\theta \, \omega^2 \wedge \omega^3. \end{aligned} \tag{A10}$$

Now we must compare this set of equations with Eqs. (A4) and pick out the $\omega^\mu{}_\nu$. The first equation of (A10), for instance, must take the form

$$d\omega^0 = -\omega^0{}_1 \wedge \omega^1 - \omega^0{}_2 \wedge \omega^2 - \omega^0{}_3 \wedge \omega^3,$$

so we guess that $\omega^0{}_1 = +f'\omega^0$. By antisymmetry we have $\omega^0{}_1 = -\omega_{01} = \omega_{10} = \omega^1{}_0$, and can verify that this choice is consistent with $0 = d\omega^1 = -\omega^1{}_0 \wedge \omega^0 + \cdots$ since $\omega^0 \wedge \omega^0 = 0$. The remaining term in $d\omega^0$ of Eqs. (A10) could arise either from $\omega^0{}_2$ or $\omega^0{}_3$, and this choice can most conveniently be settled later, so we proceed to the $d\omega^2$ equation. From $d\omega^2 = -\omega^2{}_0 \wedge \omega^0 - \omega^2{}_1 \wedge \omega^1 - \omega^2{}_3 \wedge \omega^3$ we guess that $\omega^2{}_1 = rf(r^2 + l^2)^{-1}\omega^2$, so that $\omega^2{}_0$ and $\omega^2{}_3$ terms must cancel here. In this manner, one proceeds to fill out the list below. As the solution $\omega^\mu{}_\nu$ of Eqs. (A4) and (A5) is known to be unique, the proof that a guess for a set of $\omega^\mu{}_\nu$ is correct is simply that it satisfies these equations. We have then

$$\begin{aligned} \omega^0{}_1 &= +\omega^1{}_0 = f'\omega^0, \\ \omega^0{}_2 &= +\omega^2{}_0 = lf(r^2 + l^2)^{-1}\omega^3, \\ \omega^0{}_3 &= +\omega^3{}_0 = -lf(r^2 + l^2)^{-1}\omega^2, \\ \omega^2{}_3 &= -\omega^3{}_2 = lf(r^2 + l^2)^{-1}\omega^0 - (r^2 + l^2)^{-\frac{1}{2}} \cot\theta \, \omega^3, \\ \omega^3{}_1 &= -\omega^1{}_3 = rf(r^2 + l^2)^{-1}\omega^3, \\ \omega^1{}_2 &= -\omega^2{}_1 = -rf(r^2 + l^2)^{-1}\omega^2. \end{aligned} \tag{A11}$$

Of a possible 24 connection components $\Gamma^\mu{}_{\alpha\beta}$, only the seven nonvanishing ones which appear via Eq. (A2) in Eqs. (A11) caused us any labor. That the others vanish we discovered by finding no need for additional terms in Eqs. (A11), not by explicitly evaluating a formula for $\Gamma^\mu{}_{\alpha\beta}$ and obtaining a zero result. The curvature computation is now purely mechanical. The first of Eqs. (A6), for instance, reads

$$\theta^0{}_1 = d\omega^0{}_1 + \omega^0{}_2 \wedge \omega^2{}_1 + \omega^0{}_3 \wedge \omega^3{}_1. \tag{A12}$$

Only two terms appear in the sum because of the antisymmetry of $\omega_{\mu\nu}$. Substituting in this and similar formulas from Eq. (A11) yields

$$\begin{aligned} \theta_{01} &= -2A\omega^0 \wedge \omega^1 - 2D\omega^2 \wedge \omega^3, \\ \theta_{02} &= +C\omega^0 \wedge \omega^2 + D\omega^3 \wedge \omega^1, \\ \theta_{03} &= +C\omega^0 \wedge \omega^3 + D\omega^1 \wedge \omega^2, \\ \theta_{23} &= +2B\omega^2 \wedge \omega^3 - 2D\omega^0 \wedge \omega^1, \\ \theta_{31} &= -C\omega^3 \wedge \omega^1 + D\omega^0 \wedge \omega^2, \\ \theta_{12} &= -C\omega^1 \wedge \omega^2 + D\omega^0 \wedge \omega^3, \end{aligned} \tag{A13}$$

where

$$\begin{aligned} A &= \tfrac{1}{4}(f^2)'', \\ B &= \tfrac{1}{2}[-f^2 + 1 + 4l^2f^2/(r^2 + l^2)](r^2 + l^2)^{-1}, \\ C &= [\tfrac{1}{2}r(f^2)' + l^2f^2/(r^2 + l^2)](r^2 + l^2)^{-1}, \\ D &= [\tfrac{1}{2}l(f^2)' - rlf^2/(r^2 + l^2)](r^2 + l^2)^{-1}. \end{aligned} \tag{A14}$$

The first line of Eqs. (A13), for example, tells us, by comparison with Eq. (A6), that $R_{0101} = -2A$, $R_{0123} = -2D$, $R_{0102} = 0$, $R_{0131} = 0$, etc. The contractions necessary to form $R_{\mu\nu}$, for instance

$$R_{11} = R_1{}^0{}_{10} + R_1{}^2{}_{12} + R_1{}^3{}_{13} = -R_{1010} + R_{1212} + R_{1313} = 2A - 2C,$$

are readily performed by scanning Eqs. (A13). Thus we find

$$R \equiv R^\mu_\mu \equiv R^{\mu\nu}{}_{\mu\nu} = 4(A + B - 2C), \tag{A15}$$

while for

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R,$$

the only nonvanishing components are

$$\begin{aligned} G_{11} &= -G_{00} = 2(C - B), \\ G_{22} &= G_{33} = 2(C - A). \end{aligned} \tag{A16}$$

The empty-space Einstein equations thus require $A = B = C$.

Each of the quantities A, B, C, and D is effectively an invariant since the basis vectors ω^0 and ω^1 we used can be characterized geometrically, while the form (A13) of the curvature tensor is invariant under rotations in the 23 plane. We can characterize the vector ω^1 as the unit normal to the orbits $r =$ const of the group of motions; similarly, the contravariant vector $\mathbf{e}_0$ from the dual basis can be characterized as the unit vector parallel to that unique Killing vector which commutes with all the Killing vectors. These four invariants reduce

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A) Geometrical objects such as vectors, tensors, etc. are concepts that are identified in terms of some chosen basis relative to which one specifies the components of these geometrical objects.

The power of the 4 fundamental equations of differential geometry is that they subsume <u>any</u> chosen basis.

However, from the perspective of mathematical calculations and understanding as well as physical significance there are only two types of bases: those induced by a chosen coordinate system and those which by virtue of a given metric are orthonormal.

B) Coordinate components vs. orthonormal, a.k.a. physical or tetrad or vierbein ("four-leg") components. The latter play a key role in the Cartan-Misner calculus of curvature.

The diffence is illustrated in the following

Example ("coordinate vs. orthonormal basis").

Consider the static spherically symmetic spacetime geometry of the Schwarzschild solution to the Einstein field equations.

This geometry is mathematized by its metric tensor:

a) Relative to the Schwarzschild coordinate induced basis $\{dt, dr, d\theta, d\varphi\}$ this metric is

$$g = "\cdot" = ds^2 = \underbrace{-\left(1-\frac{2m}{r}\right)}_{\mathbb{e}_t\cdot\mathbb{e}_t = g_{tt}} dt\otimes dt + \underbrace{\frac{1}{1-\frac{2m}{r}}}_{\mathbb{e}_r\cdot\mathbb{e}_r = g_{rr}} dr\otimes dr + \underbrace{r^2}_{\mathbb{e}_\theta\cdot\mathbb{e}_\theta = g_{\theta\theta}} d\theta\otimes d\theta + \underbrace{r^2\sin^2\theta}_{\mathbb{e}_\varphi\cdot\mathbb{e}_\varphi = g_{\varphi\varphi}} d\varphi\otimes d\varphi$$

b) Relative to the orthonormal (physical) basis $\{\hat\omega^t, \hat\omega^r, \hat\omega^\theta, \hat\omega^\varphi\}$ this metric is

(38.7)

$$g = \text{“}\cdot\text{”} = ds^2 = \quad -\hat\omega^t \otimes \hat\omega^t \quad + \hat\omega^r \otimes \hat\omega^r + \hat\omega^\theta \otimes \hat\omega^\theta + \hat\omega^\varphi \otimes \hat\omega^\varphi$$

c) The relation between the two is

$$\hat\omega^t = \left|1 - \frac{2m}{r}\right|^{1/2} dt$$

$$\hat\omega^r = \frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}} dr$$

$$\hat\omega^\theta = \quad r \quad d\theta$$

$$\hat\omega^\varphi = r\sin\theta \; d\varphi$$

d) The coordinate vector basis dual to $\{dt, dr, d\theta, d\varphi\}$ is

$$\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\right\} \equiv \{\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}.$$

It satisfies the duality relation $\langle dx^\mu | \frac{\partial}{\partial x^\nu}\rangle = \delta^\mu{}_\nu$.

e) The orthonormal vector basis dual to $\{\hat\omega^t, \hat\omega^r, \hat\omega^\theta, \hat\omega^\varphi\}$ is

$$\left\{
\begin{aligned}
&\frac{\mathbf{e}_t}{|\mathbf{e}_t\cdot\mathbf{e}_t|^{1/2}} = \frac{\mathbf{e}_t}{|g_{tt}|^{1/2}} = \frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}}\frac{\partial}{\partial t} \equiv \hat{\mathbf{e}}_t \\
&\frac{\mathbf{e}_r}{|\mathbf{e}_r\cdot\mathbf{e}_r|^{1/2}} = \frac{\mathbf{e}_r}{|g_{rr}|^{1/2}} = \left|1-\frac{2m}{r}\right|^{1/2}\frac{\partial}{\partial r} \equiv \hat{\mathbf{e}}_r \\
&\frac{\mathbf{e}_\theta}{|\mathbf{e}_\theta\cdot\mathbf{e}_\theta|^{1/2}} = \frac{\mathbf{e}_\theta}{|g_{rr}|^{1/2}} = \frac{1}{r}\frac{\partial}{\partial \theta} \equiv \hat{\mathbf{e}}_\theta \\
&\frac{\mathbf{e}_\varphi}{|\mathbf{e}_\varphi\cdot\mathbf{e}_\varphi|^{1/2}} = \frac{\mathbf{e}_\theta}{|g_{\theta\theta}|^{1/2}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \equiv \hat{\mathbf{e}}_\varphi
\end{aligned}
\right\} \quad \langle \hat\omega^\mu | \hat{\mathbf{e}}_\nu\rangle = \delta^\mu{}_\nu$$

f) Coordinate basis expansion and orthonormal basis expansion of a vector

$$\mathbf{u} = \hat u^\mu \hat{\mathbf{e}}_\mu = u^\mu \mathbf{e}_\mu = \sum_{\mu=0}^{3} \underbrace{u^\mu |g_{\mu\mu}|^{1/2}}_{\hat u^\mu} \underbrace{\frac{1}{|g_{\mu\mu}|^{1/2}}\mathbf{e}_\mu}_{\hat{\mathbf{e}}_\mu}$$

(38.8)

$$\underline{u} = u^t \frac{\partial}{\partial t} + u^r \frac{\partial}{\partial r} + u^\theta \frac{\partial}{\partial \theta} + u^\varphi \frac{\partial}{\partial \varphi}$$

$$= \underbrace{u^t \left|1-\tfrac{2m}{r}\right|^{1/2}}_{\hat{u}^t} \underbrace{\frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}} e_t}_{\hat{e}_t} + \underbrace{u^r \frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}}}_{\hat{u}^r} \underbrace{\left|1-\tfrac{2m}{r}\right|^{1/2} e_r}_{\hat{e}_r} + \underbrace{u^\theta r}_{\hat{u}^\theta} \underbrace{\tfrac{1}{r} e_\theta}_{\hat{e}_\theta} + \underbrace{u^\varphi r\sin\theta}_{\hat{u}^\varphi} \underbrace{\frac{1}{r\sin\theta} e_\varphi}_{\hat{e}_\varphi}$$

Thus

$$\hat{u}^\mu = u^\mu |g_{\mu\mu}|^{1/2}$$

$$\hat{u}^t = u^t \left|1-\tfrac{2m}{r}\right|^{1/2} \qquad \hat{u}_t = -\hat{u}^t$$

$$\hat{u}^r = u^r \frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}} \qquad \hat{u}_r = \hat{u}^r$$

$$\hat{u}^\theta = u^\theta\, r \qquad \hat{u}_\theta = \hat{u}^\theta$$

$$\hat{u}^\varphi = u^\varphi\, r\sin\theta \qquad \hat{u}_\varphi = \hat{u}^\varphi$$

g) Coordinate basis expansion and orthonormal basis expansion of a covector

Consider the covector potential $\underset{\sim}{A} = A_t\, dt + A_r\, dr + A_\theta\, d\theta + A_\varphi\, d\varphi$

$$\underset{\sim}{A} = \hat{A}_\mu \hat{\omega}^\mu = A_\mu\, dx^\mu = \sum_{\mu=0}^{3} \underbrace{A_\mu \frac{1}{|g_{\mu\mu}|^{1/2}}}_{\hat{A}_\mu} \underbrace{|g_{\mu\mu}|^{1/2} dx^\mu}_{\hat{\omega}^\mu}$$

The linear independence of the basis elements implies

$$\hat{A}_t = \frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}} A_t\,; \quad A_t = \left|1-\tfrac{2m}{r}\right|^{1/2} \hat{A}_t\,; \qquad \hat{A}^t = -\hat{A}_t$$

$$\hat{A}_r = \left|1-\tfrac{2m}{r}\right|^{1/2} A_r\,; \quad A_r = \frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}} \hat{A}_r\,; \qquad \hat{A}_r = \hat{A}^r$$

$$\hat{A}_\theta = \frac{1}{r} A_\theta\,; \quad A_\theta = r\, \hat{A}_\theta\,; \qquad \hat{A}_\theta = A^\theta$$

$$\hat{A}_\varphi = \frac{1}{r\sin\theta} A_\varphi\,; \quad A_\varphi = r\sin\theta\, \hat{A}_\varphi\,; \qquad \hat{A}_\varphi = A^\varphi$$

Apply these relation to each index of $R_{\alpha\beta\gamma\delta}$.

$$R_{tr\theta\varphi} = \left|1-\tfrac{2m}{r}\right|^{1/2} \frac{1}{\left|1-\frac{2m}{r}\right|^{1/2}} (r)(r\sin\theta)\, \hat{R}_{tr\theta\varphi}$$