LECTURE 39

I. Curvature of a two-sphere

II. Ricci tensor and the curvature invariant

III. Coordinate vs. orthonormal basis components

I. To illustrate the calculation of metric compatible curvature via the Cartan-Misner method of differential forms, consider a two-sphere of radius a. The method is a three step process. It starts with the metric.

1. For the given metric of a sphere,

$$g = ds^2 = a^2\, d\theta^2 + a^2 \sin^2\theta\, d\varphi^2 \; ; \quad [g_{\mu\nu}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{bmatrix}$$

introduce an orthonormal basis of covectors:

$$g = ds^2 = (a\, d\theta)^2 + (a \sin\theta\, d\varphi)^2$$
$$= (\hat{\omega}^\theta)^2 + (\hat{\omega}^\varphi) \quad ;$$

This basis is

$$\boxed{\begin{aligned} \hat{\omega}^\theta &= a\, d\theta \\ \hat{\omega}^\varphi &= a \sin\theta\, d\varphi \end{aligned}}$$

Nota bene.

(i) From the dual basis one infers that the orthonormal basis is

$$\left\{ \hat{e}_\theta = \frac{1}{a}\frac{\partial}{\partial\theta} \equiv \frac{1}{a} e_\theta \; , \; \hat{e}_\varphi = \frac{1}{a\sin\theta}\frac{\partial}{\partial\varphi} \equiv \frac{1}{a\sin\theta}\, e_\varphi \right\}$$

and hence

$$\left[\hat{g}_{\mu\nu}\right] = \left[\hat{e}_\mu \cdot \hat{e}_\nu\right] = \left[\delta_{\mu\nu}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

(ii) The starting point for the Cartan-Misner method is the dual basis.

2. Calculate the connection one-forms by using

$$d\hat{\omega}^\mu + \hat{\omega}^\mu{}_\nu \wedge \hat{\omega}^\nu = 0 \qquad (1^{st} \text{ structure equation})$$

$$\hat{\omega}_{\mu\nu} + \hat{\omega}_{\nu\mu} = d\,\hat{g}_{\mu\nu} \qquad (\text{metric compatibility})$$

The metric coefficients $\hat{g}_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$ are constants relative to an orthonormal basis. Thus, (39.3)

$$d\hat{g}_{\mu\nu} = 0.$$

The system of linear equations for the connection one-forms are

$$\hat{\omega}^\nu \wedge \hat{\omega}^\mu{}_\nu = d\hat{\omega}^\mu$$

$$\hat{\omega}_{\mu\nu} = -\hat{\omega}_{\nu\mu}$$

(a) $$d\hat{\omega}^\theta = 0$$

$$d\hat{\omega}^\varphi = d(a\sin\theta\, d\varphi) = a\cos\theta\, d\theta\wedge$$

$$= a\cos\theta\, d\theta\wedge d\varphi$$

$$= \hat{\omega}^\theta \wedge \frac{\cos\theta}{a\sin\theta}\hat{\omega}^\varphi$$

(b)

(i) $$d\hat{\omega}^\theta + \hat{\omega}^\theta{}_\theta \wedge \hat{\omega}^\theta + \hat{\omega}^\theta{}_\varphi \wedge \hat{\omega}^\varphi = 0$$

$$\boxed{\hat{\omega}^\theta{}_\theta = \hat{\omega}_{\theta\theta} = 0}$$

$$\therefore\quad \hat{\omega}^\theta{}_\varphi \wedge \hat{\omega}^\varphi = 0$$

$$\hat{\omega}^\theta{}_\varphi = (?)\,\omega^\varphi$$

(ii) $$d\hat{\omega}^\varphi + \hat{\omega}^\varphi{}_\theta \wedge \hat{\omega}^\theta + \hat{\omega}^\varphi{}_\varphi \wedge \hat{\omega}^\varphi = 0$$

$$\boxed{\hat{\omega}^\varphi{}_\varphi = \hat{\omega}_{\varphi\varphi} = 0}$$

$$\hat{\omega}^\theta \wedge \hat{\omega}^\varphi{}_\theta = \frac{\cot\theta}{a}\hat{\omega}^\theta \wedge \hat{\omega}^\varphi$$

$$\therefore\quad \hat{\omega}^\varphi{}_\theta = \frac{\cot\theta}{a}\hat{\omega}^\varphi + (??)\,\hat{\omega}^\theta$$

Compare this with b)(ii). This is possible because

$$\hat{\omega}^\varphi{}_\theta = \hat{\omega}_{\varphi\theta} = -\hat{\omega}_{\theta\varphi} = -\hat{\omega}^\theta{}_\varphi$$

Thus

$$\omega^\varphi{}_\theta = \frac{\cot\theta}{a}\hat{\omega}^\varphi + (??)\,\hat{\omega}^\theta = -(?)\,\omega^\varphi$$

$$(??) = 0$$

$$(?) \quad = -\frac{\cot\theta}{a}$$

$$\boxed{\begin{aligned}\hat{\omega}^{\varphi}{}_{\theta} &= \frac{\cot\theta}{a}\,\hat{\omega}^{\varphi}\\ \hat{\omega}^{\theta}{}_{\varphi} &= -\frac{\cot\theta}{a}\,\hat{\omega}^{\varphi}\end{aligned}}$$

c) Conclusion

$$\left[\hat{\omega}^{\mu}{}_{\nu}\right] = \begin{bmatrix} \hat{\omega}^{\theta}{}_{\theta} = 0 & \hat{\omega}^{\theta}{}_{\varphi} = -\frac{\cot\theta}{a}\hat{\omega}^{\varphi} \\ \hat{\omega}^{\varphi}{}_{\theta} = \frac{\cot\theta}{a}\hat{\omega}^{\varphi} & \hat{\omega}^{\varphi}{}_{\varphi} = 0 \end{bmatrix}$$

③ Calculate curvature 2-form

$$\hat{\Omega}^{\mu}{}_{\nu} = d\hat{\omega}^{\mu}{}_{\nu} + \hat{\omega}^{\mu}{}_{\gamma}\wedge\hat{\omega}^{\gamma}{}_{\nu}$$

$$(i) \quad \hat{\Omega}^{\theta}{}_{\varphi} = d\hat{\omega}^{\theta}{}_{\varphi} + \hat{\omega}^{\theta}{}_{\theta}\wedge\omega^{\theta}{}_{\varphi} + \hat{\omega}^{\theta}{}_{\varphi}\wedge\omega^{\varphi}{}_{\varphi}$$

$$= d\left(-\frac{\cot\theta}{a}\hat{\omega}^{\varphi}\right) + 0 + 0$$

Nota bene.

One could use the product rule to calculate this differential:

$$d\left(-\frac{\cot\theta}{a}\hat{\omega}^{\varphi}\right) = -d\left(\frac{\cot\theta}{a}\right)\wedge\hat{\omega}^{\varphi} - \frac{\cot\theta}{a}\,d\left(\hat{\omega}^{\varphi}\right)$$

However, it is easier to revert to the coordinate basis, take the differential, and then reintroduce the orthonormal basis after that.

With that in mind, the calculation yields

$$\hat{\Omega}^{\theta}{}_{\varphi} = -d\left(\frac{1}{a}\frac{\cos\theta}{\sin\theta}\,a\sin\theta\,d\varphi\right)$$

$$= -d\left(\cos\theta\,d\varphi\right)$$

$$= \sin\theta \, d\theta \wedge d\varphi$$

$$= \frac{1}{a^2} \hat{\omega}^\theta \wedge \hat{\omega}^\varphi$$

(ii) Calculate the remaining components in the same way:

$$\hat{\Omega}^\varphi{}_\theta = d\hat{\omega}^\varphi{}_\theta + \hat{\omega}^\varphi{}_\theta \wedge \hat{\omega}^\theta{}_\theta + \hat{\omega}^\varphi{}_\varphi \wedge \omega^\varphi{}_\theta$$

$$= d\left(\frac{\cot\theta}{a} \omega^\varphi\right)$$

$$= \frac{-1}{a^2} \hat{\omega}^\theta \wedge \omega^\varphi$$

(iii) $\hat{\Omega}^\theta{}_\theta = d\hat{\omega}^\theta{}_\theta + \hat{\omega}^\theta{}_\theta \wedge \hat{\omega}^\theta{}_\theta + \hat{\omega}^\theta{}_\varphi \wedge \hat{\omega}^\varphi{}_\theta$

$$= 0 + 0 + 0$$

$$\hat{\Omega}^\varphi{}_\varphi = 0$$

(iv) Use the metric induced antisymmetry in the curvature as a computational check

$$\hat{\Omega}^\theta{}_\varphi = \hat{\Omega}_{\theta\varphi} \underset{\uparrow}{=} -\hat{\Omega}_{\varphi\theta} = -\hat{\Omega}^\varphi{}_\theta \,,$$

(!)

which agrees with the calculations.

Conclusion

$$\left[\hat{\Omega}^\mu{}_\nu\right] = \begin{bmatrix} \hat{\Omega}^\theta{}_\theta = 0 & \hat{\Omega}^\theta{}_\varphi = \frac{1}{a^2}\hat{\omega}^\theta \wedge \omega^\varphi \\ \hat{\Omega}^\varphi{}_\theta = \frac{-1}{a}\hat{\omega}^\theta \wedge \omega^\varphi & \hat{\Omega}^\varphi{}_\varphi = 0 \end{bmatrix}$$

Remark.

The hatted elements are the coefficients of $\hat{e}_\mu \otimes \hat{\omega}^\nu \hat{\Omega}^\mu{}_\nu$ relative to a physical, i.e. orthonormal, basis for each tangent space $T_p(S^2)$ on the two-sphere. For such a basis the rotational symmetry of S^2 implies that the curvature is independent of θ and φ on S^2.

Read out the non-zero components of the curvature tensor from the curvature 2-form (the 2[nd] structure equation):

$$\hat{\Omega}^{\mu}{}_{\nu} = \hat{R}^{\mu}{}_{\nu|\alpha\beta|}\,\hat{\omega}^{\alpha}\wedge\hat{\omega}^{\beta} = \hat{R}^{\mu}{}_{\nu\theta\varphi}\,\hat{\omega}^{\theta}\wedge\omega^{\varphi}$$

$$= \tfrac{1}{2}\hat{R}^{\mu}{}_{\nu\alpha\beta}\,\hat{\omega}^{\alpha}\wedge\hat{\omega}^{\beta} = \tfrac{1}{2}\left(\hat{R}^{\mu}{}_{\nu\theta\varphi} - \hat{R}^{\mu}{}_{\nu\varphi\theta}\right)\hat{\omega}^{\theta}\wedge\hat{\omega}^{\varphi}$$

$$\hat{\Omega}^{\theta}{}_{\varphi} = \tfrac{1}{2}\hat{R}^{\theta}{}_{\varphi\theta\varphi}\,\hat{\omega}^{\theta}\wedge\hat{\omega}^{\varphi} + \tfrac{1}{2}\hat{R}^{\theta}{}_{\varphi\varphi\theta}\,\hat{\omega}^{\varphi}\wedge\hat{\omega}^{\theta}$$

$$\frac{1}{a^2}\,\omega^{\theta}\wedge\omega^{\varphi} = R^{\theta}{}_{\varphi\theta\varphi}\,\hat{\omega}^{\theta}\wedge\omega^{\varphi}$$

$$\therefore \quad \boxed{R^{\theta}{}_{\varphi\theta\varphi} = \frac{1}{a^2}}$$

From the antisymmetry of $R_{\mu\nu\alpha\beta}$ under the interchange $\mu\leftrightarrow\nu$ this is the only independent non-zero component of the curvature tensor.

II Ricci tensor and curvature invariant

The curvature tensor is one of rank $\binom{1}{3}$. The rank of such a tensor is lowered by the contraction map highlighted on page 16.8 of Lecture 18. The rult is the Ricci tensor, which is of rank $\binom{0}{2}$:

$$R^{\beta}{}_{\alpha\beta\gamma} \equiv R_{\alpha\gamma}.$$

This tensor is symmetric, $R_{\alpha\gamma} = R_{\gamma\alpha}$. This is a consequence of

$$R^{\delta}{}_{\alpha\beta\gamma} + R^{\delta}{}_{\beta\gamma\alpha} + R^{\delta}{}_{\gamma\alpha\beta} = T^{\delta}{}_{\alpha\beta;\gamma} + T^{\delta}{}_{\beta\gamma;\alpha} + T^{\delta}{}_{\gamma\alpha;\beta}$$

whenever the torsion-based right hand side vanished.

For the two-sphere the Ricci tensor components are

$$\hat{R}_{\theta\theta} = \hat{R}^{\theta}{}_{\theta\theta\theta} + \hat{R}^{\varphi}{}_{\theta\varphi\theta} = \frac{1}{a^2}$$

$$\hat{R}_{\theta\varphi} = \hat{R}^{\theta}{}_{\theta\theta\varphi} + \hat{R}^{\varphi}{}_{\theta\varphi\varphi} = 0$$

$$\hat{R}_{\varphi\theta} = \hat{R}^{\theta}{}_{\varphi\theta\theta} + \hat{R}^{\varphi}{}_{\varphi\varphi\theta} = 0$$

$$\hat{R}_{\varphi\varphi} = \hat{R}^{\theta}{}_{\varphi\theta\varphi} + \hat{R}^{\varphi}{}_{\varphi\varphi\varphi} = \frac{1}{a^2}$$

$$\boxed{\left[\hat{R}_{\mu\nu}\right] = \begin{bmatrix} \hat{R}_{\theta\theta} & \hat{R}_{\theta\varphi} \\ \hat{R}_{\varphi\theta} & \hat{R}_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}}$$

The curvature invariant for the two-sphere is

$$R = \hat{R}^{\theta}{}_{\theta} + \hat{R}^{\varphi}{}_{\varphi} = \frac{2}{a^2}$$

III. Coordinate vs orthonormal basis components

The coordinate basis components of the curvature tensor are determined from its o.n. basis components by following the reasoning process as illustrated with the 1-form

$$\underset{\sim}{A} = \hat{A}_\mu \hat{\omega}^\mu = \sum_\mu \underbrace{\hat{A}_\mu |g_{\mu\mu}|^{1/2}}_{A_\mu} dx^\mu = \sum A_\mu dx^\mu .$$

Thus

$$\hat{A}_\theta \hat{\omega}^\theta + \hat{A}_\varphi \hat{\omega}^\varphi = \hat{A}_\theta \, a \, d\theta + \hat{A}_\varphi \, a \sin\theta \, d\varphi$$

implies

$$\boxed{\begin{aligned} A_\theta &= a \hat{A}_\theta \\ A_\varphi &= a \sin\theta \, \hat{A}_\varphi \, . \end{aligned}}$$

Apply this line of reasoning to each of curvature and Ricci tensor components. The result is

$$R_{\theta\varphi\theta\varphi} = (a)(a\sin\theta)(a)(a\sin\theta)\hat{R}_{\theta\varphi\theta\varphi}$$
$$= a \sin^2\theta$$

$$R_{\theta\theta} = (a)(a)\hat{R}_{\theta\theta} = 1$$

$$R_{\varphi\varphi} = (a\sin\theta)(a\sin\theta)\hat{R}_{\varphi\varphi} = \sin^2\theta$$

$$\boxed{\left[\dot{R}_{\mu\nu}\right] = \begin{bmatrix} R_{\theta\theta} & R_{\theta\varphi} \\ R_{\varphi\theta} & R_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}}$$

which is obviously different from

$$\boxed{\left[\hat{R}_{\mu\nu}\right] = \begin{bmatrix} \hat{R}_{\theta\theta} & \hat{R}_{\theta\varphi} \\ \hat{R}_{\varphi\theta} & \hat{R}_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}}\ .$$