LECTURE 9

I. *Uniformly accelerated frames*

Thomas Precession

TW [P169-174, 1st edition]

MTW [Exercise 6,9]

II. *Orthonomal basis for a family of instantaneous Lorentz frames.*

MTW [Section 6.4]

Any accelerated frame as a parametrized family of Lorentz frames is the key by which linear mathematics opens the door to non-linear mathematics. The physics of accelerated frames is a metaphysical (i.e. pertaining to reality) driving force behind this key.

I. Uniform rotationally accelerated frames.

Uniformly accelerated frames are the conceptual archetypes. Besides linearly accelerated ones, there are also uniformly rotating ones, namely those where the spatial velocity is not collinear but rather perpendicular to acceleration.

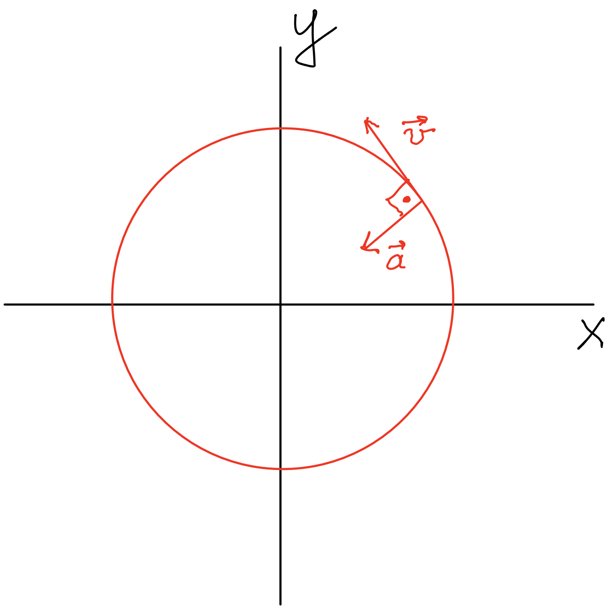


Figure 9.1a: Spatial orbit of a body.

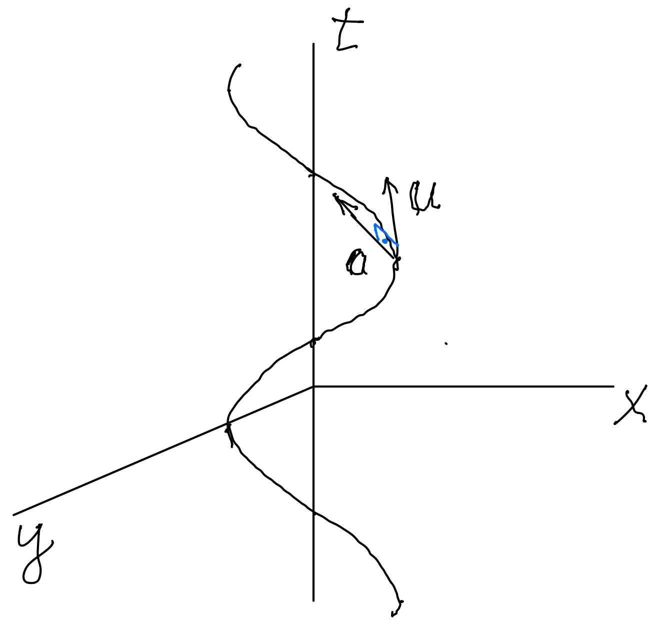


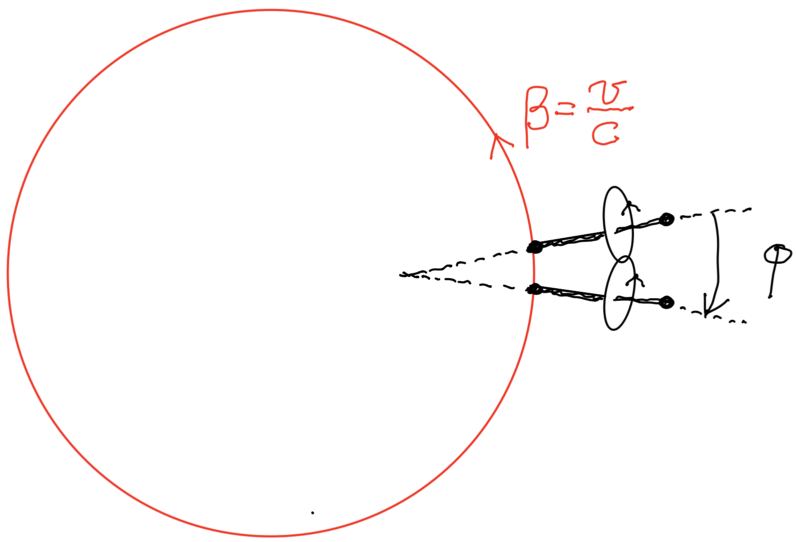
Figure 9.1b: Spacetime spiral world line of that body.

The special relativity analysis of such an acceleration lead to an explanation of the explanation of the Thomas precession of a gyroscope in a circular orbit.

Consider a freely gimbaled gyroscope attached to an accelerated frame whose x-axis is kept aligned with the direction of the gyroscope.

Suppose this frame moves along a circle with constant speed relative to the laboratory.

Q: What the orientation of the gyroscope, as measured in the Lab, after completing one circular counter clockwise circular path?



A: (a) The precession angle is

$$\varphi = 2\pi(\gamma - 1) \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

(b) The precession rate is clockwise,

$$\omega = \frac{\varphi}{T} = \Omega(\gamma - 1).$$

II. Thomas precession via relativistic reasoning.

"Relativistic reasoning" is a line of thought that is centered around the relativity of simultaneity, around the fact that simultaneity is a Lorentz frame-dependent phenomenon.

Thomas precession is a consequence of the relativity of simultaneity applied to transverse motion.

Consider a pair of meter rods A and B which are secured to gyroscopes A and B

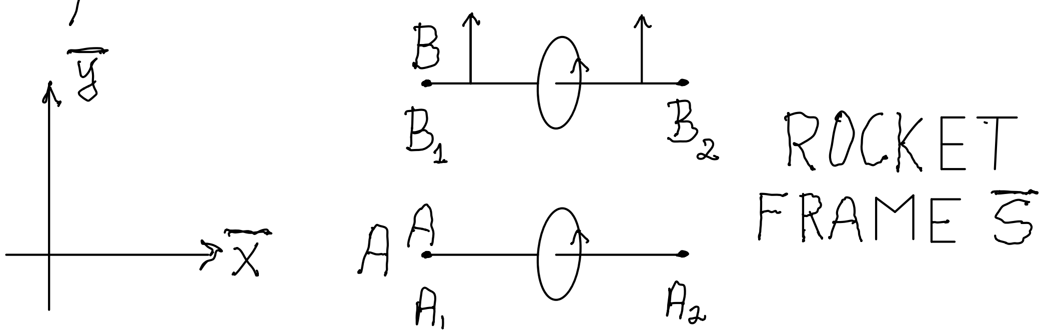


Figure 9.2: Two gyroscopes, A and B, parallel to each other but one drifting slowly relative to the other. Events B_1 and B_2 are <u>simultaneous</u>; so are A_1 and A_2.

and which are (i) parallel in the Rocket Frame

but (ii) B is drifting slowly upward (+y) in that frame

What is at first sight surprising is that relative to the LAB FRAME S, which is moving into the negative $\bar{x}$-direction, these two gyroscopes are <u>not</u> parallel to each other.

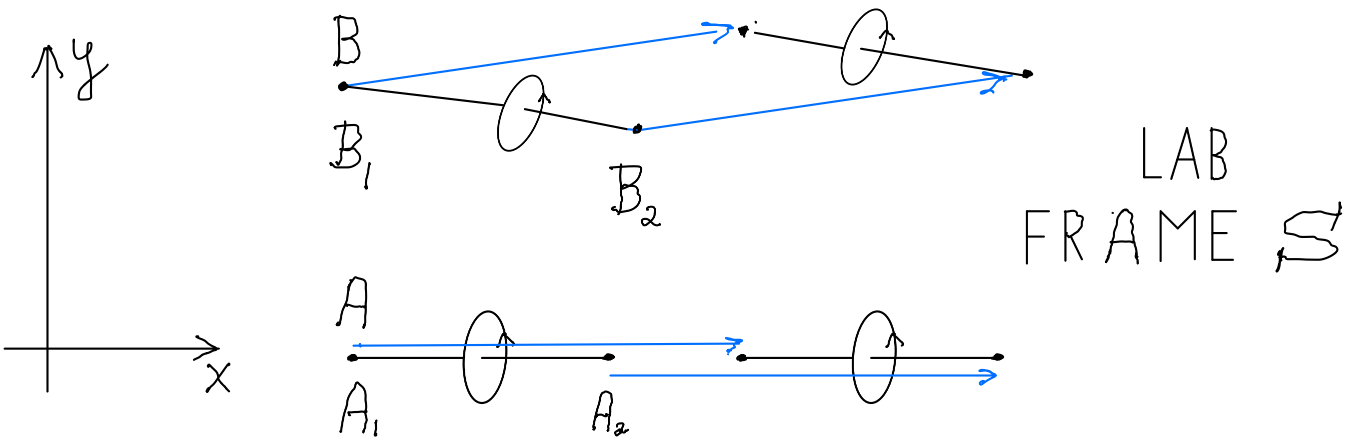
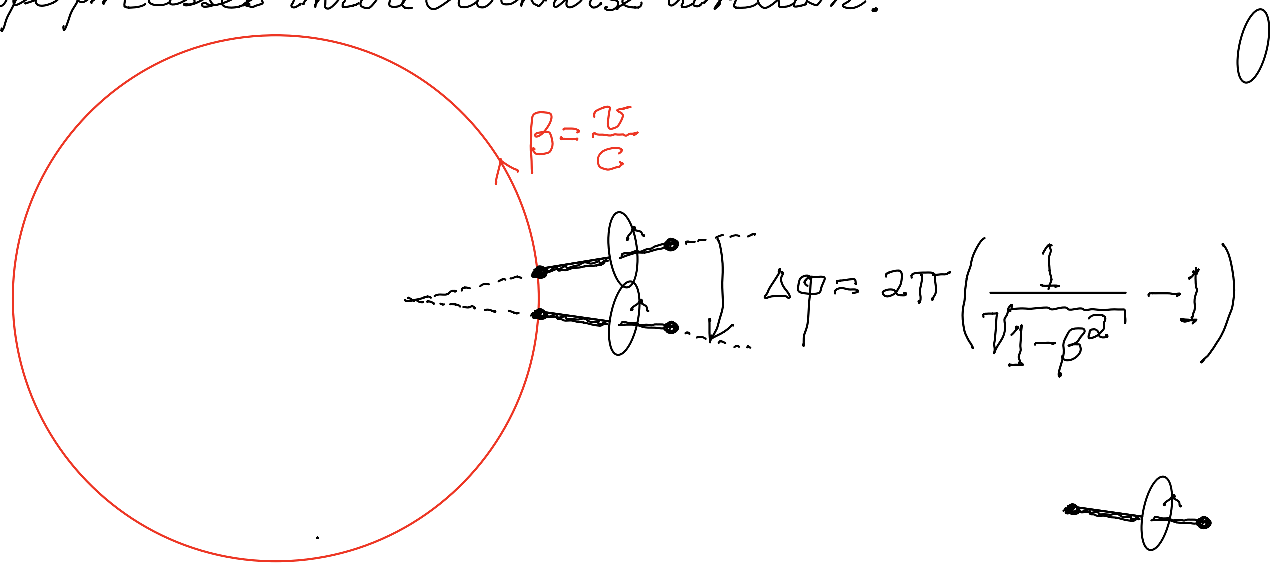


Figure 9.3: In the LAB FRAME S, whose origin start out at $t=0$ half way between A_1 and A_2 is moving towards the flash of light from A_1 and away from the flash emitted from A_2. Thus, relative to the LAB frame A_1 occurs earlier than A_2. The same holds for B_1 and B_2. (This is because B's drift vertical drift velocity is very small.) In the LAB frame B_1 occurred before B_2. Thus the tail end B_1 drifts further away from the x-axis than the tail end B_2.
Thus gyroscope B is no longer parallel to A. B will have rotated clockwise by an angle which is proportional to the vertical drift velocity.

Conclusion

Compare the orientations of a counter clockwise orbiting gyroscope at successive locations on a circular orbit. One finds that the gyroscope precesses into a clockwise direction.



$$\Delta\varphi = 2\pi\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)$$

Figure 9.4: Thomas precession of a gyroscope in a closed circular orbit.

II. Instantaneous Lorentz frames

Observations of the physical world leads us to consider two kinds of of uniform accelerative motions, linear and rotational.

Each of them is a family of instantaneous Lorentz frames mathematized by a tetrad of 4-vectors.

At each point event of the hyperbolic spacetime worldline of a linear uniformly accelerated frame,

$$\mathcal{X}(\tau): \quad x^0(\tau) = \frac{1}{g}\,\mathrm{sh}\, g\tau$$

$$x^1(\tau) = \frac{1}{g}\,\mathrm{ch}\, g\tau$$

$$x^2(\tau) = 0$$

$$x^3(\tau) = 0 \quad ,$$

this tetrad of 4-vectors is based on it 4-velocity and its 4-acceleration:

$$\left\{ \mathbf{u}(\tau) = \frac{d\mathcal{X}}{d\tau},\ \frac{\mathbf{a}(\tau)}{g(\tau)} = \frac{1}{g}\frac{d\mathbf{u}}{d\tau} \right\}: \mathbf{u}\cdot\mathbf{u} = -1,\ \mathbf{a}\cdot\mathbf{a} = g^2,\ \mathbf{a}\cdot\mathbf{u} = 0$$

Each instantaneous Lorentz frame consists of four spacetime vectors whose reprentations relative to the LAB frame are

$$(\mathbf{u})^\mu \equiv (\mathbf{e}_{0'})^\mu = (\mathrm{ch}\, g\tau,\ \mathrm{sh}\, g\tau,\ 0,\ 0)$$

$$\left(\frac{\mathbf{a}}{g}\right)^\mu \equiv (\mathbf{e}_{1'})^\mu = (\mathrm{sh}\, g\tau,\ \mathrm{ch}\, g\tau,\ 0,\ 0)$$

$$(\mathbf{e}_{2'})^\mu = (\ 0,\ \ 0,\ 1,\ 0)$$

$$(\mathbf{e}_{3'})^\mu = (\ 0,\ \ 0,\ 0,\ 1)$$

NOTATION:

The primed subscripts in the tetrad of vectors $\boxed{e_{0'}(\tau), e_{1'}(\tau), e_{2'}(\tau), e_{3'}(\tau)}$ serve as a reminder that they make up a moving frame, as compared to $\boxed{e_0, e_1, e_2, e_3}$:

$$e_0 : (e_0)^\mu = (1, 0, 0, 0)$$
$$e_1 : (e_1)^\mu = (0, 1, 0, 0)$$
$$e_2 : (e_2)^\mu = (0, 0, 1, 0)$$
$$e_3 : (e_3)^\mu = (0, 0, 0, 1)$$

which would be stationary in the inertial LAB frame, typically fixed relative to the "fixed" stars.

The superscripts (μ: 0,1,2,3) refer, of course, to the coordinate components relative to the chosen or given Lorentz frame basis.