LECTURE 9

I. *Uniformly accelerated frames*

Thomas Precession

TW [P169–174, 1st edition]

MTW [Exercise 6.9]

II. *Orthonomal basis for a family of instantaneous Lorentz frames.*

MTW [Section 6.4]

Any accelerated frame as a parametrized family of Lorentz frames is the key by which linear mathematics opens the door to non-linear mathematics. The physics of accelerated frames is a metaphysical (i.e. pertaining to reality) driving force behind this key.

I. Uniform rotationally accelerated frames.

Uniformly accelerated frames are the conceptual archetypes. Besides linearly accelerated ones, there are also uniformly rotating ones, namely those where the spatial velocity is not collinear but rather perpendicular to acceleration.

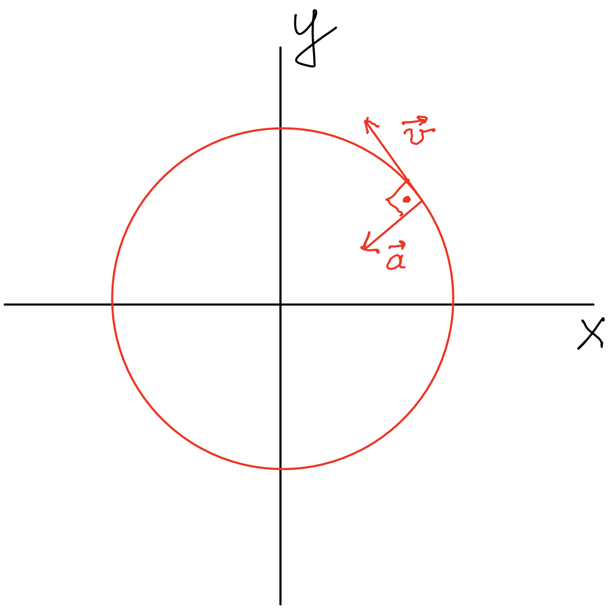


Figure 9.1a: Spatial orbit of a body.

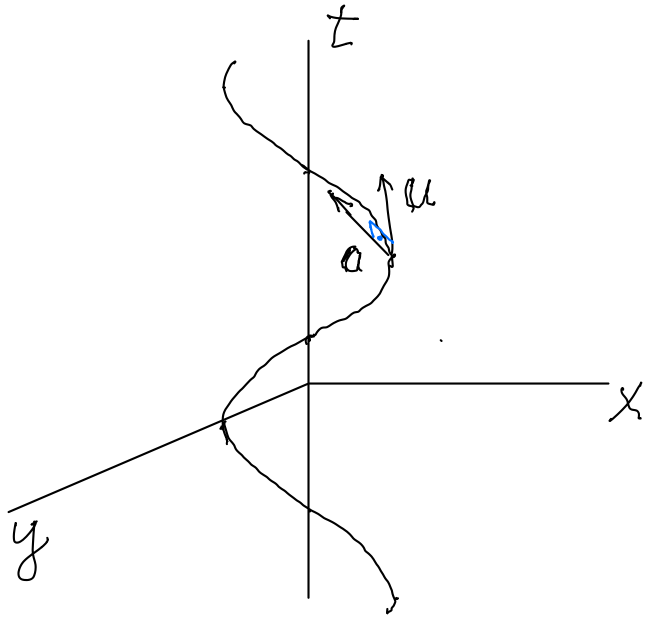


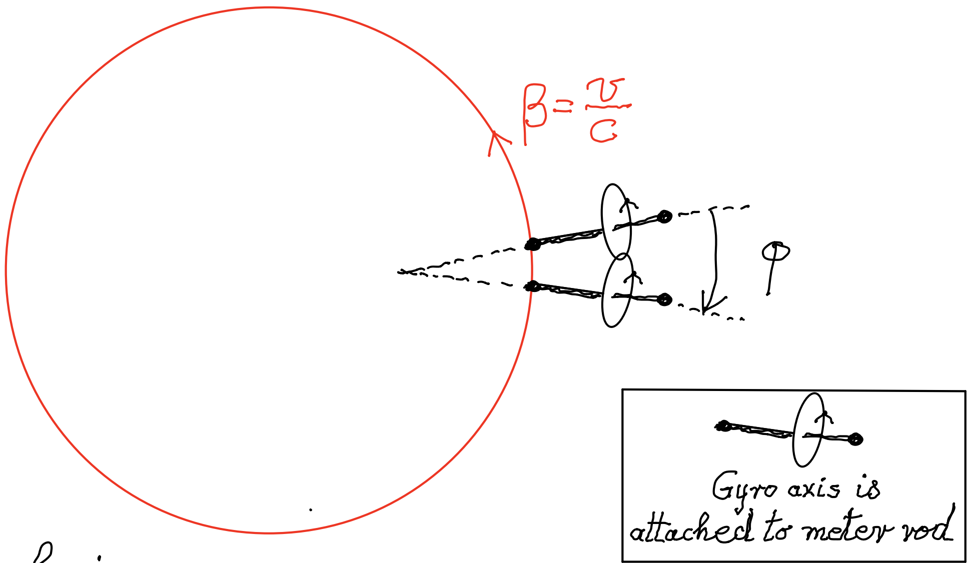
Figure 9.1b: Spacetime spiral world line of that body.

The special relativity analysis of such an acceleration lead to an explanation of the explanation of the Thomas precession of a gyroscope in a circular orbit.

Consider a freely gimbaled gyroscope attached to an accelerated frame whose x-axis is kept aligned with the direction of the gyroscope.

Suppose this frame moves along a circle with constant speed relative to the laboratory.

Q: What the orientation of the gyroscope, as measured in the Lab, after completing one circular counter clockwise circular path?



A: (a) The precession angle is

$$\varphi = 2\pi(\gamma - 1) \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

(b) The precession rate is clockwise,

$$\omega = \frac{\varphi}{T} = \Omega(\gamma - 1).$$

Thomas precession is a consequence of the relativity of simultaneity applied to transverse motion.

The context of Thomas precession is uniform circular motion, which is a type of change which physicists and mathematicians need to mathematize in order to understand it. Newton, more so than any other physicist/mathematician, made change understandable by conceptually discretizing what superficially looked like continuous change. Following his lead, we shall discretize the uniform circle into a regular polygon, each of whose sides subtends the angle $\frac{2\pi}{n}$ and then let $n \to \infty$. With $n=8$ the circle is approximated by an octagon. The adjacency of two sides implies that as $\frac{2\pi}{n} \to 0$, the velocity change of the gyroscope in transiting between these two sides is correspondingly small.

To understand Thomas precession one must compare this small change as observed by the Rocket observer with what is observed in the Lab frame.

The gyroscope has two distinct vectorial attributes in the x-y plane:

(i) The projection of its spin direction into the Lab's x-y plane and

(ii) its velocity along each side of the octagon.

the velocity as it moves along side A of the octagon before the motion is along side B with an upward tilt in the x-y plane.

These vectors are depicted in Figure 9, 2 below.

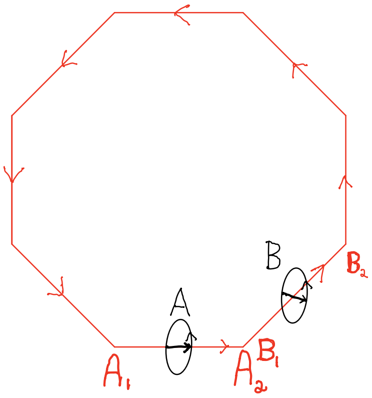


Figure 9.2: Uniform circular motion discretized as motion around an octagon with constant speed along each side.

To grasp the essence of the precession process, the gyro axis of A (black arrow) is taken to be parallel to $\overline{A_1 A_2}$. The subsequent motion of the gyro is along segment $\overline{B_1 B_2}$, but the gyro's axis B (black arrow) is no longer parallel to that of A.

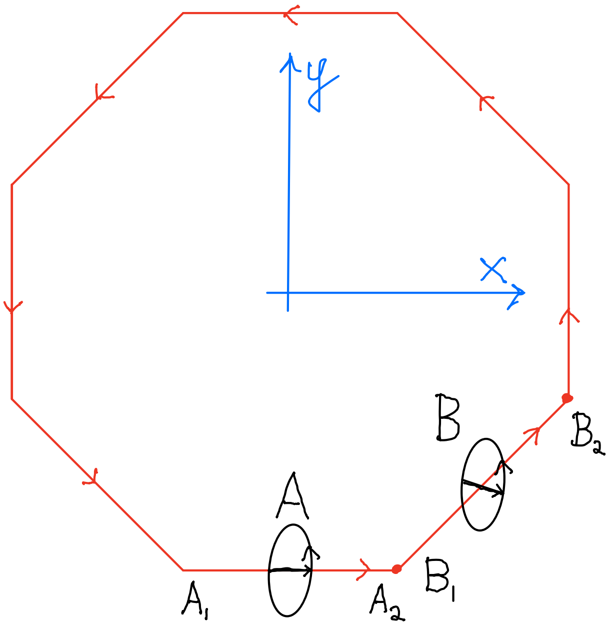


Figure 9.3a: Successive gyro motions A and B relative to the Lab frame

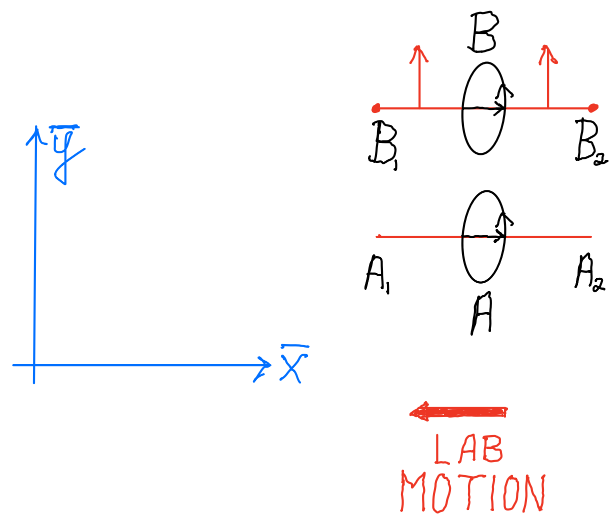


Figure 9.3b: Successive gyro motions A and B relative to the Comoving Rocket frame. Events B_1 and B_2 are simultaneous in this frame.

Compare the freely gimbaled gyroscope's orientation on side A with its orientation on B. Relative to the Comoving Rocket frame the orientation of the gyroscope at A is the same as at B: The gyro axes at A and B are parallel.

The only difference is that at B the gyro is moving with neglegibly small, but non-zero, velocity upward from A.

Events B_1 and B_2 (e.g. two lightning flashes) are simultaneous in the Rocket from. However, in the Lab frame B_1 occurred earlier, before B_2. Consequently, in its upward motion B_1 will have moved upward, away from A, further than B_2, i.e. in the Lab $\overline{B_1B_2}$ will have rotated clockwise relative to A.

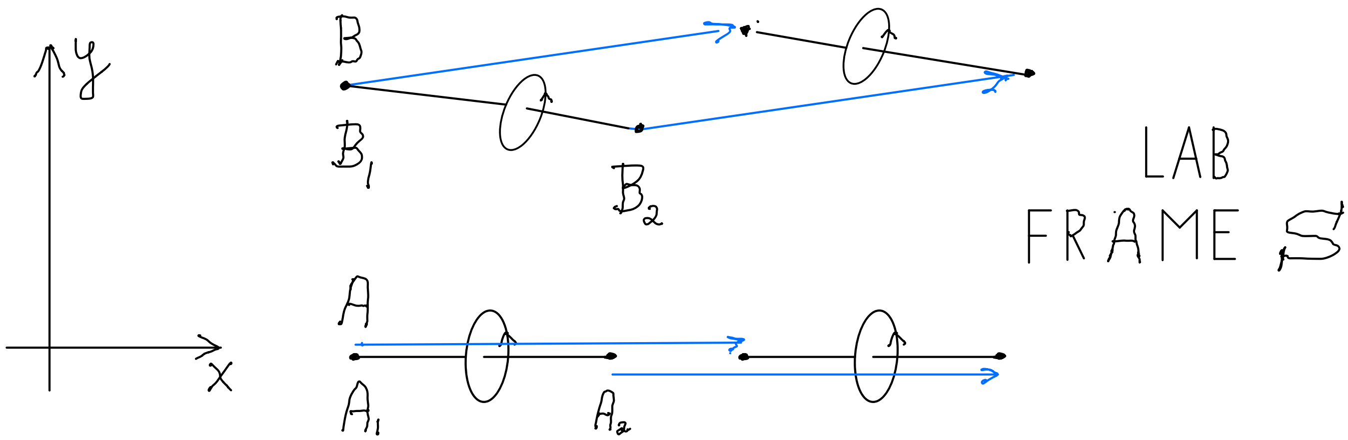


Figure 9.3: In the LAB FRAME S, whose origin starts out at $t=0$ half way between A_1 and A_2, is moving towards the flash of light from A_1 and away from the flash emitted from A_2. Thus, relative to the LAB frame A_1 occurs earlier than A_2. The same holds for B_1 and

B_2. (This is because B's ~~drift~~ vertical drift velocity is very small.) In the LAB frame B_1 occurred earlier before B_2. Thus the tail end B_1 drifts further away from the x-axis than the tail end B_2. Thus gyroscope B is no longer parallel to A. B will have rotated clockwise by an angle which is proportional to the vertical drift velocity.

The total sum of such rotation angles adds up to the total Thomas precession angle.

CONCLUSION:

Compare the orientations of a counter clockwise orbiting gyroscope at successive locations on a circular orbit. One finds that the gyroscope precesses into a clockwise direction.

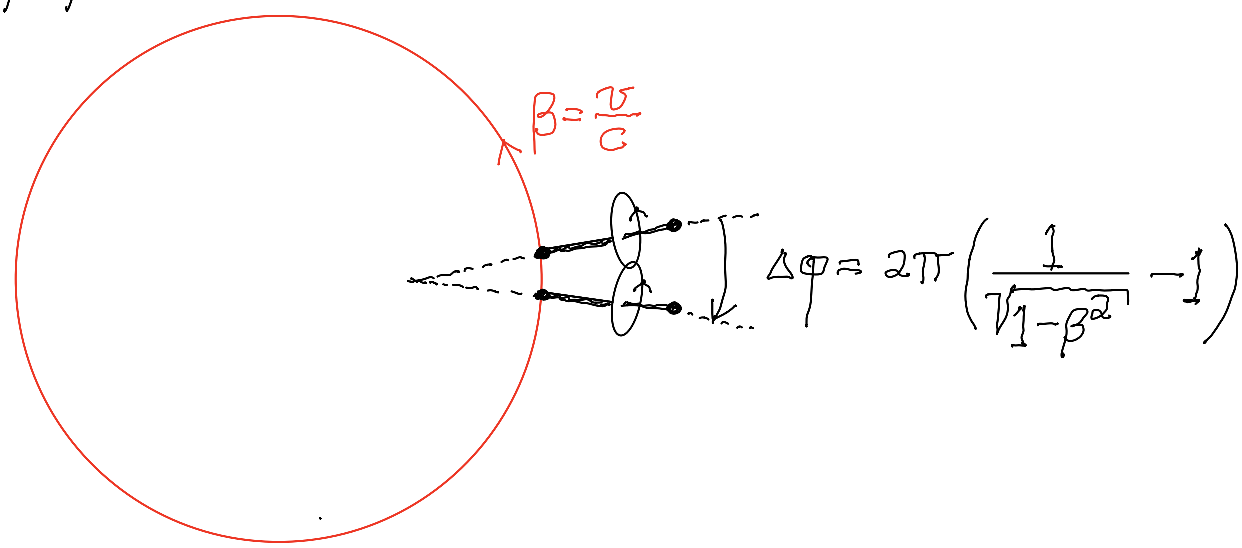


Figure 9.4: Thomas precession of a gyroscope in a closed circular orbit.

II. Instantaneous Lorentz frames

Observations of the physical world leads us to consider two kinds of of uniform accelerative motions, linear and rotational.

Each of them is a family of instantaneous Lorentz frames mathematized by a tetrad of 4-vectors.

At each point event of the hyperbolic spacetime worldline of a linear uniformly accelerated frame,

$$\mathbb{X}(\tau): \quad x^0(\tau) = \frac{1}{g}\,sh\,g\tau$$
$$x^1(\tau) = \frac{1}{g}\,ch\,g\tau$$
$$x^2(\tau) = 0$$
$$x^3(\tau) = 0 \quad ,$$

this tetrad of 4-vectors is based on its 4-velocity and its 4-acceleration:

$$\left\{\mathbb{u}(\tau) = \frac{d\mathbb{X}}{d\tau}, \frac{\mathbb{a}(\tau)}{g(\tau)} = \frac{1}{g}\frac{d\mathbb{u}}{d\tau}\right\}: \mathbb{u}\cdot\mathbb{u} = -1, \ \mathbb{a}\cdot\mathbb{a} = g^2, \ \mathbb{a}\cdot\mathbb{u} = 0$$

Each instantaneous Lorentz frame consists of four spacetime vectors whose repentations relative to the LAB frame are

$$(\mathbb{u})^\mu \equiv (\mathbb{e}_{0'})^\mu = (ch\,g\tau, sh\,g\tau, 0, 0)$$
$$\left(\frac{\mathbb{a}}{g}\right)^\mu \equiv (\mathbb{e}_{1'})^\mu = (sh\,g\tau, ch\,g\tau, 0, 0)$$
$$(\mathbb{e}_{2'})^\mu = (\ 0, \ 0, 1, 0)$$
$$(\mathbb{e}_{3'})^\mu = (\ 0, \ 0, 0, 1)$$

NOTATION:

The primed subscripts in the tetrad of vectors $\boxed{\mathbb{e}_{0'}(\tau), \mathbb{e}_{1'}(\tau), \mathbb{e}_{2'}(\tau), \mathbb{e}_{3'}(\tau)}$ serve as a reminder that they make up a moving frame, as compared to $\boxed{\mathbb{e}_0, \mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3}$:

$$\mathbb{e}_0 : (\mathbb{e}_0)^\mu = (1, 0, 0, 0)$$
$$\mathbb{e}_1 : (\mathbb{e}_1)^\mu = (0, 1, 0, 0)$$
$$\mathbb{e}_2 : (\mathbb{e}_2)^\mu = (0, 0, 1, 0)$$
$$\mathbb{e}_3 : (\mathbb{e}_3)^\mu = (0, 0, 0, 1)$$

which would be stationary in the inertial LAB frame, typically fixed relative to the "fixed" stars.

The superscripts ($\mu : 0,1,2,3$) refer, of course, to the coordinate components relative to the chosen or given Lorentz frame basis.