CHAPTER FIVE

MANIFOLDS

5.1 DIFFERENTIABLE MANIFOLDS

Definition. A *locally Euclidean space* X of dimension n is a Hausdorff topological space such that, for each $x \in X$, there exists a homeomorphism φ_x mapping some open set containing x onto an open set in R^n.

Remark. We may, if we wish, choose each φ_x so that $\varphi_x(x) = 0$ and so that the image of φ_x is a ball $B_0(\varepsilon)$. Given any φ_x homeomorphically mapping an open set U about x onto an open set in R^n, let $\varepsilon > 0$ be such that $B_{\varphi(x)}(\varepsilon) \subset \varphi_x(U)$. Let

$$\psi: B_{\varphi(x)}(\varepsilon) \to B_0(\varepsilon)$$

be translation by $-\varphi(x)$. Then

$$\tilde{\varphi}_x = \psi \circ \varphi_x|_{\varphi_x^{-1}(B_{\varphi(x)}(\varepsilon))}$$

maps $\varphi_x^{-1}(B_{\varphi(x)}(\varepsilon))$ homeomorphically onto $B_0(\varepsilon)$.

Example 1. R^n is locally Euclidean. For each $x \in R^n$, take φ_x to be the identity map.

Example 2. S^n is locally Euclidean. Given $x \in S^n$, let $y \in S^n$, $y \neq x$. Then φ_x = stereographic projection from y maps $S^n - \{y\}$ homeomorphically onto R^n.

Example 3. Projective space P^n, that is, the space of all lines through 0 in R^{n+1}, is locally Euclidean. For since P^n is covered by S^n, each $x \in P^n$ is contained in an open set homeomorphic to an open set in S^n that itself contains, about each of its points, an open set homeomorphic to an open set in R^n.

Example 4. Each open subset U of a locally Euclidean space X is locally Euclidean. For if $x \in U$, let ψ_x be a homeomorphism mapping an open set about x in X onto an open set in R^n. Take $\varphi_x = \psi_x|_{U \cap \text{domain } \psi_x}$.

Example 5. The set of all non-singular $k \times k$ matrices forms a locally Euclidean space of dimension k^2. Each $k \times k$ matrix may be identified with a k^2-tuple by stringing out the rows in a line. The non-singular matrices then form an open set of R^{k^2}, namely $\Delta^{-1}(R^1 - \{0\})$ where $\Delta: R^{k^2} \to R^1$ is the determinant function.

Definition. A *C^k-differentiable manifold* of dimension n is a pair (X, Φ) where X is a Hausdorff topological space, and Φ is a collection of maps such that the following conditions hold. (See Fig. 5.1.)

(1) $\{\text{domain } \varphi\}_{\varphi \in \Phi}$ is an open covering of X,

(2) each $\varphi \in \Phi$ maps its domain homeomorphically onto an open set in R^n,

(3) for each $\varphi, \psi \in \Phi$ with $(\text{domain } \varphi) \cap (\text{domain } \psi) \neq \emptyset$, the map $\psi \circ \varphi^{-1}$ is a C^k-map from $\varphi(\text{domain } \varphi \cap \text{domain } \psi) \subset R^n$ into R^n,

(4) Φ is maximal relative to (2) and (3); that is, if ψ is any homeomorphism mapping an open set in X onto an open set in R^n such that, for each $\varphi \in \Phi$ with domain $\varphi \cap$ domain $\psi \neq \emptyset$, $\psi \circ \varphi^{-1}$ and $\varphi \circ \psi^{-1}$ are C^k-maps from

$$\varphi(\text{domain } \varphi \cap \text{domain } \psi) \text{ and } \psi(\text{domain } \varphi \cap \text{domain } \psi)$$

into R^n—then $\psi \in \Phi$.

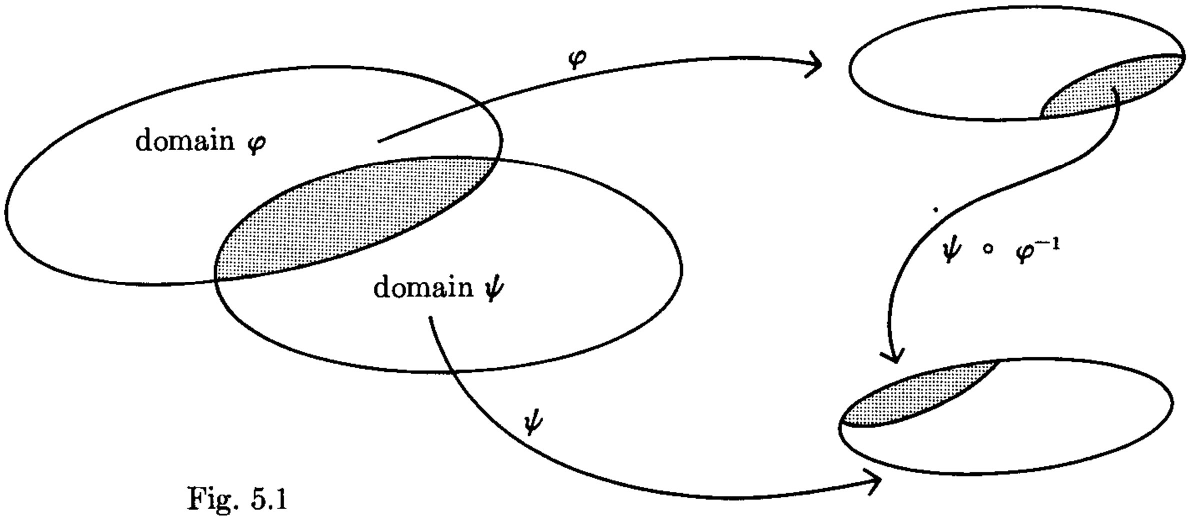

Fig. 5.1

Here k may be $0, 1, 2, \ldots, \infty, \omega$. C^0 means continuous. C^k for k finite means all partial derivatives of order less than or equal to k exist and are continuous. C^∞ means all partial derivatives of all orders exist and are continuous. C^ω means real analytic; that is, the function may be expressed as a convergent Taylor series in a neighborhood of each point.

Note that a C^k-manifold is a locally Euclidean space and a locally Euclidean space gives rise to a C^0-manifold.

If $n = 2$ and, in Condition (3), "C^k" is replaced by "complex analytic" (where R^2 is identified with the complex numbers C^1), (X, Φ) is called a *complex analytic manifold* of complex dimension 1 or a *Riemann surface*. Φ is then called a *complex structure* or *conformal structure* on X.

The maps $\varphi \in \Phi$ are called *coordinate systems*. More precisely, the map $\varphi \in \Phi$ is called a *coordinate system on the open set* (domain φ) $\subset X$. For $x \in X$, a *coordinate system about* x is a coordinate system $\varphi \in \Phi$ such that $x \in$ domain φ.

Remark. Each of the above Examples 1, 2, 3, and 5 of locally Euclidean spaces form the underlying space of a C^∞-manifold. You need only check that the maps φ_x satisfy Condition (3) for a manifold, and then take Φ to be a maximal set containing $\{\varphi_x\}_{x \in X}$. Example 4 above also carries over to manifolds. Namely, if (X, Φ) is a C^k-manifold and U is an open set in X, then $(U, \Phi|_U)$ is a C^k-manifold, where $\Phi|_U = \{\varphi|_U\}_{\varphi \in \Phi}$.

Definitions. Let (X, Φ) be a C^k-manifold. A real-valued function $f: X \to R^1$ is a *C^s-function* $(s \leq k)$, denoted $f \in C^s(X, R^1)$, if, for each $\varphi \in \Phi$, $f \circ \varphi^{-1}$ is a C^s-function mapping the image of $\varphi \subset R^n$ into R^1.

Let (X, Φ) be a C^k-manifold, and let $x \in X$. A real-valued function f is said to be *of class C^s $(s \leq k)$ in a neighborhood of* x, denoted $f \in C^s(X, x, R^1)$, if

$$U = (\text{domain } f)$$

is an open set in X containing x, and $f \in C^s(U, R^1)$, where U has the C^k-manifold structure as an open set in X.

Remarks. Note that we are able to define C^s-functions on X because (1) X looks locally (via the coordinate systems $\varphi \in \Phi$) like R^n, and we know what it means for a function on R^n to be C^s; and (2) if U = domain φ and V = domain ψ for $\varphi, \psi \in \Phi$, with

$U \cap V \neq \emptyset$, the concept of a C^s-function in a neighborhood of x in $U \cap V$ is the same relative to the coordinate system φ as to the coordinate system ψ, because $\psi \circ \varphi^{-1}$ is a C^k-homeomorphism and $k \geq s$.

Note also that if f and g are C^s-functions in a neighborhood of x, then $f + g$ and fg (product) are C^s-functions in a neighborhood of x, where

$$\text{domain } (f + g) = \text{domain } (fg) = (\text{domain } f) \cap (\text{domain } g).$$

Definition. Let (X, Φ) be a C^k-manifold, and let $\varphi \in \Phi$ be a coordinate system on U = domain φ. Let r_j: $R^n \to R^1$ be the j-th coordinate function on R^n; that is, $r_j(a_1, a_2, \ldots, a_n) = a_j$ for $(a_1, \ldots, a_n) \in R^n$. The j-th *coordinate function* of the coordinate system φ is the function x_j: $U \to R^1$ defined by $x_j = r_j \circ \varphi$.

Remark. x_j: $U \to R^1$ is a C^k-function. The n-tuple of functions $(x_1, \ldots, x_n)$ is sometimes also referred to as a coordinate system.

Definition. Let (X_1, Φ_1) and (X_2, Φ_2) be C^k-manifolds (not necessarily of the same dimension). A mapping Ψ: $X_1 \to X_2$ is of class C^s $(s \leq k)$, denoted $\Psi \in C^s(X_1, X_2)$, if, whenever $f \in C^s(X_2, R^1)$, then $f \circ \Psi \in C^s(X_1, R^1)$.

Exercise 1. Show that, if Ψ: $X_1 \to X_2$ is of class C^s $(s \geq 0)$, then Ψ is continuous.

Remarks. We shall confine our attention to C^∞-manifolds. This will include, in particular, C^ω-manifolds and complex analytic manifolds of dimension 1. We shall use the word "smooth" to denote C^∞.

We now proceed to define the concept of tangent vector on a manifold. Recall that, in Euclidean space, a vector at a point defines a map which sends each smooth function into a real number, namely, the directional derivative with respect to the given vector. Moreover, the vector is determined by its values on all smooth functions. We shall use this property to define tangent vectors on a manifold.

Definition. Let (X, Φ) be a smooth manifold and let $x \in X$. A *tangent vector* at x is a map v: $C^\infty(X, x, R^1) \to R^1$ such that, if φ is a (fixed) coordinate system with $x \in U$ = domain φ, then there exists an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers with the following property. For each $f \in C^\infty(X, x, R^1)$,

$$v(f) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i} (f \circ \varphi^{-1})|_{\varphi(x)}.$$

(Note that if W = domain f, then φ and f are both defined on the open set $U \cap W$ containing x, so that $f \circ \varphi^{-1}$ is a smooth function with domain $\varphi(U \cap W) \subset R^n$ containing $\varphi(x)$.)

Remark. If v: $C^\infty(X, x, R^1) \to R^1$ has the property required above of a tangent vector with respect to one coordinate system φ about x, then it also has this property with respect to any other coordinate system about x. For, if ψ is another such coordinate system, then, using the chain rule,

$$\begin{aligned} v(f) &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i} (f \circ \varphi^{-1})|_{\varphi(x)} \\ &= \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i} (f \circ \psi^{-1} \circ \psi \circ \varphi^{-1})|_{\varphi(x)} \\ &= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \frac{\partial}{\partial r_j} (f \circ \psi^{-1})|_{\psi(x)} \, J_{ji}(\psi \circ \varphi^{-1})|_{\varphi(x)}, \end{aligned}$$

where $J_{ij}(\psi \circ \varphi^{-1})$ is the Jacobian matrix of the function $\psi \circ \varphi^{-1}$. Hence

$$v(f) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_i J_{ji}(\psi \circ \varphi^{-1})|_{\varphi(x)} \right) \frac{\partial}{\partial r_j} (f \circ \psi^{-1})|_{\psi(x)}.$$

Setting

$$b_j = \sum_{i=1}^{n} a_i J_{ji}(\psi \circ \varphi^{-1})|_{\varphi(x)},$$

we obtain

$$v(f) = \sum_{j=1}^{n} b_j \frac{\partial}{\partial r_j} (f \circ \psi^{-1})|_{\psi(x)}.$$

Thus, to check if v is a tangent vector at x, it suffices to check the required property in any one coordinate system at x.

Notation. Given a coordinate system φ about x, let $x_j = r_j \circ \varphi$ denote the j-th coordinate function of φ. By $\partial/\partial x_j$ $(j = 1, \ldots, n)$ is meant the tangent vector at x defined by

$$\frac{\partial}{\partial x_j}(f) = \frac{\partial}{\partial r_j}(f \circ \varphi^{-1})|_{\varphi(x)}$$

for $f \in C^\infty(X, x, R^1)$. Thus $\partial/\partial x_j$ corresponds, relative to the coordinate system φ, to the n-tuple $(0, 0, \ldots, 1, \ldots, 0)$, where the 1 is in the j-th spot.

Remark 1. If $x_1, \ldots, x_n$ are the coordinate functions of a coordinate system φ about x, and $y_1, \ldots, y_n$ are those of a coordinate system ψ about x, then the above computation shows that

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}(y_i) \frac{\partial}{\partial y_i}.$$

Remark 2. A tangent vector v at $x \in X$ has the following properties. For any $f, g \in C^\infty(X, x, R^1)$ and for $\lambda \in R^1$,

(1) $v(f + g) = v(f) + v(g)$

(2) $v(\lambda f) = \lambda v(f)$

(3) $v(fg) = v(f)g(x) + f(x)v(g)$.

These three properties say that the map v: $C^\infty(X, x, R^1) \to R^1$ is a *derivation*. Moreover, these properties characterize tangent vectors; that is, we could have defined a tangent vector to be a map v: $C^\infty(X, x, R^1) \to R^1$ satisfying (1)-(3) above, and then proved that, relative to any coordinate system φ about x, $v = \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ for some n-tuple $(a_1, \ldots, a_n)$ of real numbers, where x_i is the i-th coordinate function of φ.

Remark 3. The set X_x of tangent vectors at x form a vector space under the following rules of addition and scalar multiplication:

$$(v_1 + v_2)(f) = v_1(f) + v_2(f) \qquad (v_1, v_2 \in X_x),$$
$$(\lambda v_1)(f) = \lambda(v_1(f)) \qquad (v_1 \in X_x, \lambda \in R^1).$$

To see that $v_1 + v_2$ and λv_1 are tangent vectors at x, let φ be a coordinate system about x, with coordinate functions $(x_1, \ldots, x_n)$. Then

$$v_1 = \sum_{i=1}^{n} a_i(\partial/\partial x_i) \quad \text{and} \quad v_2 = \sum_{i=1}^{n} b_i(\partial/\partial x_i)$$

for some $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. It is then easy to check that

$$v_1 + v_2 = \sum_{i=1}^{n} (a_i + b_i) \frac{\partial}{\partial x_i},$$
$$\lambda v_1 = \sum_{i=1}^{n} (\lambda a_i) \frac{\partial}{\partial x_i}.$$

The map $(a_1, \ldots, a_n) \to \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ gives a vector space isomorphism $R^n \to X_x$, so X_x has dimension n. Moreover, it is clear that $\{\partial/\partial x_i\}_{i \in \{1, \ldots, n\}}$ is a basis for X_x. The space X_x is called the *tangent space* to X at x. It is also denoted by $T(X)_x$ or by $T(X, x)$.

For φ and ψ two coordinate systems at x, with coordinate functions $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively, the formula

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}(y_i) \frac{\partial}{\partial y_i}$$

merely expresses the vector $\partial/\partial x_j$ in terms of the basis $\{\partial/\partial y_i\}_{i \in \{1, \ldots, n\}}$. Thus the change of basis matrix from the basis $\{\partial/\partial y_i\}$ of X_x to the basis $\{\partial/\partial x_i\}$ is precisely the Jacobian matrix $((\partial/\partial x_j)(y_i))$.

Remark 4. The tangent space $T(R^n, a)$ to R^n at a point $a \in R^n$ is naturally isomorphic with R^n itself. The isomorphism $R^n \to T(R^n, a)$ is given by

$$(\lambda_1, \ldots, \lambda_n) \to \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial r_i}.$$

Notation. We shall henceforth omit the Φ from our notation for a differentiable manifold (X, Φ). To be sure, a locally Euclidean space X may have two or more distinct differentiable structures on it (or it may have none), but we shall denote a manifold (X, Φ) merely by X and shall assume that a definite differentiable structure is given on it.

Definition. Let X and Y be smooth manifolds. Let Ψ: $X \to Y$ be a smooth map. The *differential* of Ψ at $x \in X$ is the map $d\Psi$: $X_x \to Y_{\Psi(x)}$ defined as follows. For $v \in X_x$ and $g \in C^\infty(Y, \Psi(x), R^1)$, $(d\Psi(v))(g) = v(g \circ \Psi)$.

Remark. It is easily checked that $d\Psi(v)$ is indeed a tangent vector at $\Psi(x)$. For, if φ is a coordinate system about x with coordinate functions $(x_1, \ldots, x_n)$, and τ is a coordinate system about $\Psi(x)$ with coordinate functions $(y_1, \ldots, y_m)$, then $v = \sum_{i=1}^{n} a_i(\partial/\partial x_i)$ for some real numbers a_i; and if $g \in C^\infty(Y, \Psi(x), R^1)$, then

$$\begin{aligned}
[d\Psi(v)](g) &= v(g \circ \Psi) = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}(g \circ \Psi) \\
&= \sum_{i=1}^{n} a_i \frac{\partial}{\partial r_i}(g \circ \tau^{-1} \circ \tau \circ \Psi \circ \varphi^{-1})|_{\varphi(x)} \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{m} \frac{\partial}{\partial s_j}(g \circ \tau^{-1})|\tau \circ \Psi(x) \frac{\partial}{\partial r_i}(s_j \circ \tau \circ \Psi \circ \varphi^{-1})|_{\varphi(x)} \\
&\qquad [(s_1, \ldots, s_m) \text{ coordinates on } R^m] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \frac{\partial}{\partial y_j}(g) \frac{\partial}{\partial x_i}(y_j \circ \Psi) \\
&= \left[\sum_{j=1}^{m} v(y_j \circ \Psi) \frac{\partial}{\partial y_j}\right](g).
\end{aligned}$$

Since this holds for all $g \in C^\infty(Y, \Psi(x), R^1)$,

$$\boxed{d\Psi(v) = \sum_{j=1}^{m} v(y_j \circ \Psi)\frac{\partial}{\partial y_j},}$$

and, in particular, $d\Psi(v)$ is a tangent vector. Furthermore, it is clear that $d\Psi$ is a linear transformation $X_x \to Y_{\Psi(x)}$. Since

$$d\Psi\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^{m} \frac{\partial}{\partial x_i}(y_j \circ \Psi)\frac{\partial}{\partial y_j},$$

this linear transformation $d\Psi$ has matrix

$$(d\Psi)_{ij} = \left(\frac{\partial}{\partial x_j}(y_i \circ \Psi)\right)$$

relative to the bases $\{\partial/\partial x_i\}_{i \in \{1,\ldots,n\}}$ and $\{\partial/\partial y_j\}_{j \in \{1,\ldots,m\}}$.

Remark. Let X, Y, and Z be smooth manifolds. Let $\Psi\colon X \to Y$ and $\Phi\colon Y \to Z$ be smooth maps. Then $d(\Phi \circ \Psi) = d\Phi \circ d\Psi$.

Proof. Suppose $v \in X_x$ and $h \in C^\infty(Z, \Phi \circ \Psi(x), R^1)$. Then

$$\begin{aligned}[d(\Phi \circ \Psi)(v)](h) &= v(h \circ (\Phi \circ \Psi)) = v((h \circ \Phi) \circ \Psi)\\ &= d\Psi(v)(h \circ \Phi)\\ &= [d\Phi(d\Psi(v))](h)\\ &= [(d\Phi \circ d\Psi)(v)](h). \qquad \square\end{aligned}$$

Remark. Let X be a smooth manifold, and let U be open in X. Then U is itself a smooth manifold. Moreover, the inclusion map $i\colon U \to X$ is a smooth map. Indeed, $f \in C^\infty(X, R^1)$ implies $f|_U \in C^\infty(U, R^1)$. Furthermore, the differential

$$di\colon\ T(U, u_0) \to T(X, u_0) \qquad (u_0 \in U)$$

is an isomorphism; we shall identify these two linear spaces.

Exercise 2. If $u_0 \in U$ an open set in X, construct a function $h \in C^\infty(X, R^1)$ such that

$$h(x) = \begin{cases}1 & (x \in W \text{ an open set containing } u_0),\\ 0 & (x \notin U).\end{cases}$$

(Hint: Make use of the smooth function $g\colon R^1 \to R^1$ defined by

$$g(t) = \begin{cases}e^{-1/t^2} & (t > 0)\\ 0 & (t \le 0).)\end{cases}$$

If $f_1 \in C^\infty(U, u_0, R^1)$, use Exercise 1 to show that there exists a smaller open set W and $f \in C^\infty(X, R^1)$ such that $f|_W = f_1|_W$.

Remark. Let X be a smooth manifold, and let $f \in C^\infty(X, R^1)$. Let us compute df. For $v \in T(X, x)$, $df(v) \in T(R^1, f(x))$. Since $T(R^1, f(x))$ is 1-dimensional, $df(v) = \lambda(d/dr)$ for some $\lambda \in R^1$. To determine λ, it suffices to evaluate $df(v)$ on the coordinate function r: $R^1 \to R^1$ as follows.

$$\lambda = \left[\lambda \frac{d}{dr}\right](r) = [df(v)](r) = v(r \circ f) = v(f).$$

Thus $df(v) = v(f)(d/dr)$. Now $T(R^1, f(x))$ is naturally isomorphic with R^1 via the isomorphism $\lambda(d/dr) \to \lambda$. Let us identify these two spaces through this isomorphism. Then df: $T(X, x) \to R^1$ is a linear functional on $T(X, x)$; that is, df is a member of the dual space $T^*(X, x)$ and is, as such, given by

$$df(v) = v(f) \qquad (v \in T(X, x)).$$

$T^*(X, x)$ is called the *cotangent space* at x.

Definition. Let X be a smooth manifold. A *smooth curve* in X is a smooth map α from some (open or closed) interval $\subset R^1$ into X. If the domain of α is a closed interval $[a, b]$, smoothness of α means that α admits a smooth extension

$$\tilde{\alpha}: (a - \varepsilon, b + \varepsilon) \to X.$$

(Note that open intervals are open sets in R^1 and hence are smooth manifolds.)

A *broken C^∞-curve* in X is a continuous map α: $[a, b] \to X$ together with a subdivision of $[a, b]$ on whose closed subintervals α is a C^∞ curve.

Example.

$$\alpha(t) = \begin{cases} (t, t \sin 1/t) & (t \in (0, 1]) \\ (0, 0) & (t = 0) \end{cases}$$

is *not* a smooth curve in R^2 because it admits no smooth extension past 0.

Definition. Let α: $I \to X$ (I an interval $\subset R^1$) be a smooth curve in X. The *tangent vector* to α at time t ($t \in I$), denoted by $\dot{\alpha}(t)$, is defined by

$$\dot{\alpha}(t) = d\tilde{\alpha}\left(\left(\frac{d}{dr}\right)_t\right).$$

Note that $\dot{\alpha}(t)$ is well defined, even at the endpoints of I.

Remark. Given a tangent vector $v \in X_x$, let α: $I \to X$ be a smooth curve whose tangent vector at time $t = 0$ is v. (Such a curve may be obtained by taking a coordinate system φ about x, finding a curve (for example, the straight line) in R^n whose tangent vector at time 0 is $d\varphi(v)$, and pulling this curve back to X by φ^{-1}.) Then, for $f \in C^\infty(X, x, R^1)$,

$$v(f) = \dot{\alpha}(0)(f) = d\tilde{\alpha}\left(\left(\frac{d}{dr}\right)_0\right)(f) = \frac{d}{dr}(f \circ \tilde{\alpha})|_0.$$

Thus $v(f)$ is the derivative of the "restriction" of f to the curve α. Moreover, two curves α_1 and α_2 have the same tangent vector v at time 0 if and only if $\alpha_1(0) = \alpha_2(0)$ and

$$\frac{d}{dr}(f \circ \tilde{\alpha}_1)|_0 = \frac{d}{dr}(f \circ \tilde{\alpha}_2)|_0$$

for all $f \in C^\infty(X, x, R^1)$. (See Fig. 5.2.) We may use this equation to define an equivalence relation on the set of all curves α with $\alpha(0) = x$. Then we get a one-to-one correspondence between equivalence classes of curves through x and tangent vectors at x. Thus, we could have defined a tangent vector at x to be such an equivalence class of curves through x.

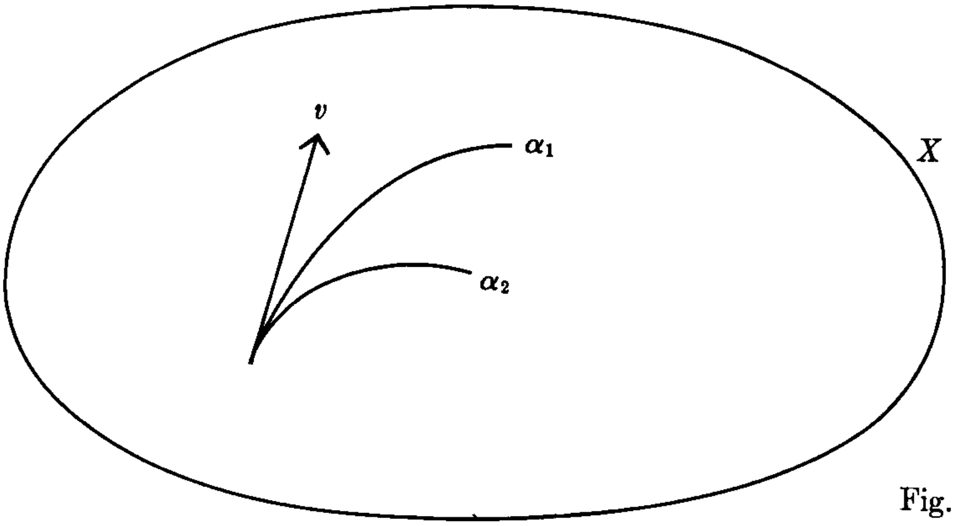

Fig. 5.2

5.2 DIFFERENTIAL FORMS

Definitions. Let X be a smooth manifold. Define

$$T(X) = \bigcup_{x \in X} T(X, x) \quad \text{and} \quad T^*(X) = \bigcup_{x \in X} T^*(X, x).$$

$T(X)$ is called the *tangent bundle* of X. $T^*(X)$ is called the *cotangent bundle* of X.

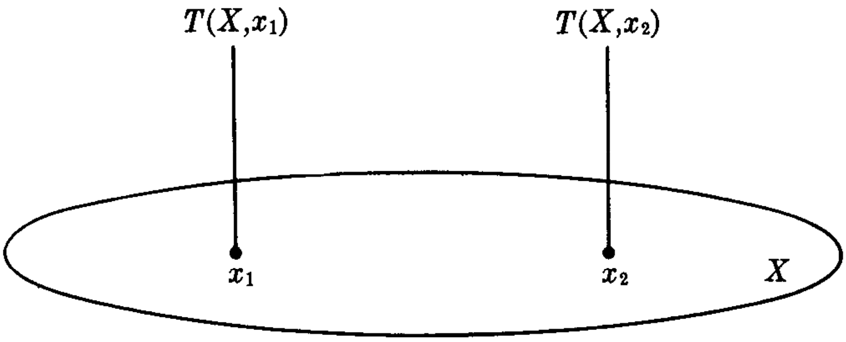

Fig. 5.3

A *projection map* π: $T(X) \to X$ is defined as follows. If $v \in T(X)$, then $v \in T(X, x)$ for some (unique) $x \in X$; set $\pi(v) = x$. Similarly, there is a projection map from $T^*(X)$ onto X that we shall also denote by π.

A *vector field* on X is a map $V: X \to T(X)$ such that $\pi \circ V = i_X$. A vector field V is *smooth* if for each $f \in C^\infty(X, R^1)$, $Vf \in C^\infty(X, R^1)$. Here Vf is defined by

$$(Vf)(x) = V(x)f.$$

A *differential* 1-*form* on X is a map $\omega: X \to T^*(X)$ such that $\pi \circ \omega = i_X$. A differential 1-form ω is *smooth* if for each smooth vector field V on X,

$$\omega(V) \in C^\infty(X, R^1).$$

Here $\omega(V)$ is defined by $(\omega(V))(x) = \omega(x)(V(x))$. We shall denote the set of all smooth vector fields on X by $C^\infty(X, T(X))$ and the set of all smooth 1-forms by $C^\infty(X, T^*(X))$.

Exercise. Define a manifold structure on $T(X)$ so that π is a smooth map and so that a vector field V is smooth if and only if it is a smooth map from $X \to T(X)$.

(Hint: for $\varphi: U \to R^n$ a local coordinate system on X, with coordinate functions $(x_1, \ldots, x_n)$, define $\tilde{\varphi}: \pi^{-1}(U) \to R^{2n}$ by

$$\tilde{\varphi}(v) = (\varphi \circ \pi(v), b_1, \ldots, b_n),$$

where $b_1, \ldots, b_n \in R^1$ are such that $v = \sum_{i=1}^{n} b_i \, \partial/\partial x_i$.)

Remark 1. Let $f \in C^\infty(X, R^1)$. Then $df \in C^\infty(X, T^*(X))$. For if $V \in C^\infty(X, T(X))$, then $df(V) = Vf \in C^\infty(X, R^1)$.

Remark 2. $C^\infty(X, T(X))$ and $C^\infty(X, T^*(X))$ are both vector spaces over the reals under the operations of pointwise addition and scalar multiplication. For example, if V_1 and $V_2 \in C^\infty(X, T(X))$, then $V_1 + V_2$ is defined by $(V_1 + V_2)(x) = V_1(x) + V_2(x)$; and if $\lambda \in R^1$, then λV_1 is defined by $(\lambda V_1)(x) = \lambda(V_1(x))$.

Remark 3. Let φ be a coordinate system on X with domain U and coordinate functions $(x_1, x_2, \ldots, x_n)$. Then the following hold.

(1) $(\partial/\partial x_i) \in C^\infty(U, T(U))$ for $i \in \{1, \ldots, n\}$. $\partial/\partial x_i$ is smooth because if

$$f \in C^\infty(U, R^1), \quad \text{then} \quad f \circ \varphi^{-1} \in C^\infty(\varphi(U), R^1),$$

and, for each $x \in U$,

$$\left[\frac{\partial}{\partial x_i}(f)\right](x) = \left[\frac{\partial}{\partial r_i}(f \circ \varphi^{-1})\right](\varphi(x)) = \left[\left[\frac{\partial}{\partial r_i}(f \circ \varphi^{-1})\right] \circ \varphi\right](x);$$

that is,

$$\frac{\partial}{\partial x_i}(f) = \left[\frac{\partial}{\partial r_i}(f \circ \varphi^{-1})\right] \circ \varphi \in C^\infty(U, R^1).$$

(2) If $V \in C^\infty(U, T(U))$, then there exist functions $a_i \in C^\infty(U, R^1)$ for $i \in \{1, \ldots, n\}$, such that $V = \sum_{i=1}^{n} a_i (\partial/\partial x_i)$. These functions a_i exist because

$$\{(\partial/\partial x_i)(x)\}_{i \in \{1, \ldots, n\}}$$

is a basis for $T(X, x)$. They are smooth because $(\partial/\partial x_i)(x_j) = \delta_{ij}$, so that

$$a_j = \sum_{i=1}^{n} a_i \delta_{ij} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}(x_j) = V(x_j) \in C^\infty(U, R^1).$$

(3) If $V \in C^\infty(X, T(X))$, then $V|_U \in C^\infty(U, T(U))$ by a previous exercise, and $V|_U = \sum_{i=1}^{n} a_i (\partial/\partial x_i)$ as in (2) with $a_i \in C^\infty(U, R^1)$.

(4) $dx_j \in C^\infty(U, T^*(U))$ for $j \in \{1, \ldots, n\}$ because $x_j \in C^\infty(U, R^1)$. Furthermore, $\{dx_j\}$ is at each point the dual basis to $\{\partial/\partial x_j\}$ because

$$dx_j\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}(x_j) = \delta_{ij}.$$

(5) If $\omega \in C^\infty(U, T^*(U))$, then there exist $a_i \in C^\infty(U, R^1)$ such that $\omega = \sum_{i=1}^{n} a_i\, dx_i$. These functions a_i exist because $\{dx_i\}$ is at each point a basis for the cotangent space. They are smooth because

$$a_i = \sum a_j\, dx_j\left(\frac{\partial}{\partial x_i}\right) = \omega\left(\frac{\partial}{\partial x_i}\right) \in C^\infty(U, R^1).$$

(6) If $f \in C^\infty(U, R^1)$, then

$$df = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(f)\, dx_j$$

because $df = \sum_{j=1}^{n} a_j\, dx_j$ for some a_j, and

$$a_i = \sum_{j=1}^{n} a_j\, dx_j\left(\frac{\partial}{\partial x_i}\right) = df\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}(f).$$

We have just seen that if $f \in C^\infty(X, R^1)$, then df is a smooth differential 1-form. We now introduce differential k-forms.

REVIEW OF EXTERIOR ALGEBRAS. Let V be an n-dimensional vector space over the reals. Then the following hold.

(1) The vector space $\Lambda^k(V^*)$ is the space of all skew-symmetric k-linear functions on V; that is, each $\tau \in \Lambda^k(V^*)$ is a map $\tau\colon \underbrace{V \oplus \cdots \oplus V}_{k\text{-times}} \to R^1$ such that for all $v_1, \ldots, v_k, v_j' \in V$, $\lambda \in R^1$,

(1) $$\tau(v_1, \ldots, v_{j-1}, v_j + v_j', v_{j+1}, \ldots, v_k) = \tau(v_1, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_k) + \tau(v_1, \ldots, v_{j-1}, v_j', v_{j+1}, \ldots, v_k);$$

(2) $$\tau(v_{\pi(1)}, \ldots, v_{\pi(k)}) = (-1)^\pi\, \tau(v_1, \ldots, v_k); \text{ and}$$

(3) $$\tau(v_1, \ldots, v_{j-1}, \lambda v_j, v_{j+1}, \ldots, v_k) = \lambda\tau(v_1, \ldots, v_j, \ldots, v_k),$$

where π is any element of the permutation group S_k on k letters, and $(-1)^\pi$ is +1 if π is an even permutation or -1 if π is an odd permutation. This second condition is equivalent to requiring that if two vectors in the argument of τ are interchanged, then the value of τ on these vectors changes sign. The dimension of $\Lambda^k(V^*)$ is equal to the binomial coefficient $\binom{n}{k}$ for $k \le n$; it is zero for $k > n$.

(2) If we set $\mathcal{G}(V^*) = \sum_{k=0}^{n} \oplus \Lambda^k(V^*)$, where $\Lambda^0(V^*) = R^1$, a product is defined on $\mathcal{G}(V^*)$ as follows. If $\tau \in \Lambda^k(V^*)$ and $\mu \in \Lambda^\ell(V^*)$, their product $\tau \wedge \mu$ is the element of $\Lambda^{k+\ell}(V^*)$ defined by

$$\tau \wedge \mu(v_1, \ldots, v_{k+\ell}) = \frac{1}{(k+\ell)!} \sum_{\pi \in S_{k+\ell}} (-1)^\pi \tau(v_{\pi(1)}, \ldots, v_{\pi(k)})\, \mu(v_{\pi(k+1)}, \ldots, v_{\pi(k+\ell)}).$$

Since $\mathcal{G}(V^*)$ is generated by such μ and τ, this multiplication extends to $\mathcal{G}(V^*)$ by linearity, that is, by requiring that exterior multiplication $\wedge$ be distributive with respect to vector addition. This multiplication is associative and $\mathcal{G}(V^*)$ is an algebra, with unit 1. However, multiplication is not commutative: if $\mu \in \Lambda^k(V^*)$ and $\tau \in \Lambda^\ell(V^*)$, then

$$\mu \wedge \tau = (-1)^{k\ell}\, \tau \wedge \mu.$$

(3) If $\varphi_1, \ldots, \varphi_n$ is a basis for V^*, then

$$[\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k};\ 1 \le i_1 < i_2 < \ldots < i_k \le n]$$

is a basis for $\Lambda^k(V^*)$. Hence the union of these sets over $k \in \{1, \ldots, n\}$, together with $1 \in \Lambda^0(V^*)$, is a basis for $\mathcal{G}(V^*)$. It follows that the dimension of $\mathcal{G}(V^*)$ is 2^n.

If $v_1, \ldots, v_k \in V$, the value of $\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k}$ on these vectors is given by

$$(\varphi_{i_1} \wedge \ldots \wedge \varphi_{i_k})(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\pi \in S_k} (-1)^\pi \varphi_{i_1}(v_{\pi(1)}) \cdots \varphi_{i_k}(v_{\pi(k)}).$$

(4) $\mathcal{G}(V^*)$ has the following properties:

(1) $1 \in \mathcal{G}(V^*)$, $\quad V^* \subset \mathcal{G}(V^*)$;

(2) $\mathcal{G}(V^*)$ is generated by 1 and V^*;

(3) $\varphi \wedge \varphi = 0$ whenever $\varphi \in V^*$; and

(4) dimension $\mathcal{G}(V^*) = 2^n$.

These properties in fact characterize $\mathcal{G}(V^*)$; that is, if $\tilde{\mathcal{G}}(V^*)$ is any algebra over the reals satisfying properties (1)-(4), then $\tilde{\mathcal{G}}(V^*)$ and $\mathcal{G}(V^*)$ are isomorphic (as algebras).

(Note that Condition (3) is equivalent to the condition that $\varphi_1 \wedge \varphi_2 = -\varphi_2 \wedge \varphi_1$ for all $\varphi_1, \varphi_2 \in V^*$.)

(5) If L: $V^* \to V^*$ is a linear transformation, then L induces a unique algebra homomorphism $\tilde{L}$: $\mathcal{G}(V^*) \to \mathcal{G}(V^*)$ which extends the map L. $\tilde{L}$ preserves degrees; that is, $\tilde{L}$: $\Lambda^k(V^*) \to \Lambda^k(V^*)$. In particular, $\tilde{L}$: $\Lambda^n(V^*) \to \Lambda^n(V^*)$. Hence, since dim $\Lambda^n(V^*) = 1$, there exists a scalar λ such that $\tilde{L}|_{\Lambda^n(V^*)} = \lambda \cdot 1$. This scalar λ is precisely $\Delta(L)$, the determinant of L.

(6) The algebra $\mathcal{G}(V^*)$ is called the *Grassmann algebra*, or *exterior algebra*, of V^*. Elements of $\mathcal{G}(V^*)$ are called *forms* on V. Forms in $\Lambda^k(V^*)$ are said to be of *degree k*.

Review of Exterior Algebras ends here.

Now let X be a smooth manifold. Let

$$\Lambda^k(X) = \bigcup_{x \in X} \Lambda^k(T^*(X, x)),$$

and let

$$\mathcal{G}(X) = \bigcup_{x \in X} \mathcal{G}(T^*(X, x)).$$

As usual, we shall denote the projection maps from these spaces onto X by π. These spaces can each be given the structure of a smooth manifold such that π is a smooth map.

Definition. A *k-form* on X is a mapping $\mu: X \to \Lambda^k(X)$ such that $\pi \circ \mu = i_X$. A k-form μ on X is *smooth* if whenever $V_1, \ldots, V_k$ are smooth vector fields on X, then

$$\mu(V_1, \ldots, V_k) \in C^\infty(X, R^1),$$

where

$$\mu(V_1, \ldots, V_k)(x) = \mu(x)(V_1(x), \ldots, V_k(x)).$$

A *differential form* on X is a mapping $\omega: X \to \mathcal{G}(X)$ such that $\pi \circ \omega = i_X$; it is *smooth* if its component in $\Lambda^k(X)$ is smooth for each k. The set of smooth k-forms on X is denoted by $C^\infty(X, \Lambda^k(X))$. The set of all smooth differential forms is denoted by $C^\infty(X, \mathcal{G}(X))$. Note that $C^\infty(X, \Lambda^k(X))$ is a vector space under pointwise addition and scalar multiplication, and that $C^\infty(X, \mathcal{G}(X))$ is an algebra under the additional operation of pointwise exterior multiplication.

Remark 1. A 0-form on X is just a real-valued function on X; it is a smooth 0-form if and only if it is a smooth function.

Remark 2. Let φ be a local coordinate system on X, with domain U and coordinate functions $(x_1, \ldots, x_n)$. Then $\{dx_1, \ldots, dx_n\}$ is a basis for $T^*(X, x)$ for each $x \in U$. Hence

$$[dx_{i_1} \wedge \ldots \wedge dx_{i_k};\ i_1 < \ldots < i_k]$$

is a basis for $\Lambda^k(T^*(X, x))$ for each $x \in U$. Thus, the restriction to U of each k-form μ on X can be expressed as

$$\mu = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k},$$

where each $a_{i_1 \ldots i_k}$ is a real-valued function on U. Furthermore, μ is smooth if and only if, for each (φ, U), $a_{i_1 \ldots i_k} \in C^\infty(U, R^1)$. This is because

$$a_{i_1 \ldots i_k} = k!\,\mu\left(\frac{\partial}{\partial x_{i_1}}, \ldots, \frac{\partial}{\partial x_{i_k}}\right).$$

THEOREM 1. **Let X be a smooth manifold. There exists a unique linear map $d: C^\infty(X, \mathcal{G}(X)) \to C^\infty(X, \mathcal{G}(X))$, called the *exterior differential*, such that the following properties hold.**

(1) $d: C^\infty(X, \Lambda^k(X)) \to C^\infty(X, \Lambda^{k+1}(X))$;

(2) $d(f) = df$ **(ordinary differential) for** $f \in C^\infty(X, \Lambda^0(X))$;

(3) if $\mu \in C^\infty(X, \Lambda^k(X))$ and $\tau \in C^\infty(X, \mathcal{G}(X))$, then

$$d(\mu \wedge \tau) = (d\mu) \wedge \tau + (-1)^k \mu \wedge d\tau; \text{ and}$$

(4) $d^2 = 0$.

Remark. For the proof we need the following lemma, which asserts that for any exterior differentiation operator d, $(d\omega)(x)$ depends only on the behavior of ω in a small neighborhood of x.

LEMMA. Let d: $C^\infty(X, \mathcal{G}(X)) \to C^\infty(X, \mathcal{G}(X))$ be linear and satisfy the conditions of the theorem. Suppose $\omega \in C^\infty(X, \mathcal{G}(X))$ is such that $\omega|_W = 0$ for some open set $W \subset X$. Then $(d\omega)|_W = 0$. Hence, if $\omega, \tau \in C^\infty(X, \mathcal{G}(X))$ are such that $\omega|_W = \tau|_W$ for some open set W, then $(d\omega)|_W = (d\tau)|_W$.

Proof. Suppose $\omega|_W = 0$. Let $x_0 \in W$. Let $f \in C^\infty(X, R^1)$ be such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \notin W$. Then $f\omega$ is identically zero on X, so that

$$0 = d(f\omega) = (df) \wedge \omega + f\, d\omega.$$

Evaluating at x_0 gives $(d\omega)(x_0) = 0$. Since this holds for all $x_0 \in W$, $d\omega|_W = 0$. If $\omega|_W = \tau|_W$, then $(\omega - \tau)|_W = 0$, so that

$$0 = [d(\omega - \tau)]|_W = [d\omega - d\tau]|_W \quad \text{and} \quad d\omega|_W = d\tau|_W. \quad \square$$

Proof of Theorem 1.

Uniqueness. Suppose d: $C^\infty(X, \mathcal{G}(X)) \to C^\infty(X, \mathcal{G}(X))$ satisfies the conditions of the theorem. Let $x \in X$, and let φ be a local coordinate system about x with domain U and coordinate functions $(x_1, \ldots, x_n)$. Let $\omega \in C^\infty(X, \Lambda^k(X))$. Then the restriction of ω to U can be expressed as

$$\omega|_U = \sum a_{i_1 \ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

for some $a_{i_1 \ldots i_k} \in C^\infty(U, R^1)$. Now the right-hand side of this equation is not a differential form on X, so we cannot apply d to it. However, let U_1 be an open ball containing x with $\bar{U}_1$ compact and $\subset U$, and let $g \in C^\infty(X, R^1)$ be such that $g(x) = 1$ for $x \in U_1$ and $g(x) = 0$ for $x \notin U$. Then $\tilde{\omega} \in C^\infty(X, \Lambda^k(X))$, where

$$\tilde{\omega} = \sum (ga_{i_1 \ldots i_k})\, d(gx_{i_1}) \wedge \ldots \wedge d(gx_{i_k}).$$

Here, by gh, for $h \in C^\infty(U, R^1)$, is meant the smooth function on X defined by

$$(gh)(x) = \begin{cases} g(x)h(x) & (x \in U) \\ 0 & (x \notin U). \end{cases}$$

Furthermore, $\tilde{\omega}|_{U_1} = \omega|_{U_1}$. By the lemma, $(d\omega)|_{U_1} = (d\tilde{\omega})|_{U_1}$. Now

$$\begin{aligned} d\tilde{\omega} &= \sum d[ga_{i_1 \ldots i_k}\, d(gx_{i_1}) \wedge \ldots \wedge d(gx_{i_k})] && \text{(by linearity)} \\ &= \sum d(ga_{i_1 \ldots i_k}) \wedge d(gx_{i_1}) \wedge \ldots \wedge d(gx_{i_k}) \\ &\quad + \sum ga_{i_1 \ldots i_k}\, d(d(gx_{i_1}) \wedge \ldots \wedge d(gx_{i_k})) && \text{(by Property (3))} \\ &= \sum d(ga_{i_1 \ldots i_k}) \wedge d(gx_{i_1}) \wedge \ldots \wedge d(gx_{i_k}), \end{aligned}$$

since each term of the second sum is zero by Properties (3) and (4). In particular, since g is identically 1 on U_1, and since $(d\omega)|_{U_1} = (d\tilde{\omega})|_{U_1}$,

$$(d\omega)|U_1 = \sum_{i_1<\ldots<i_k} \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(a_{i_1\ldots i_k})\, dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Thus if d exists, its value at x on k-forms must be given by this formula. Since x was arbitrary in X, and since every differential form is a sum of k-forms, $k \in \{0, 1, \ldots, n\}$, uniqueness is established.

Existence. We first define d locally. Let φ be a local coordinate system on X with domain U and coordinate functions $(x_1, \ldots, x_n)$. (Note that U is itself a smooth manifold.) Define d_U: $C^\infty(U, \mathcal{G}(U)) \to C^\infty(U, \mathcal{G}(U))$ as follows. For

$$\omega = \sum a_{i_1\ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \in C^\infty(U, \Lambda^k(U)),$$

define

$$\boxed{d_U\omega = \sum \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(a_{i_1\ldots i_k})\, dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.}$$

Extend d_U to $C^\infty(U, \mathcal{G}(U))$ by linearity. Then Properties (1) and (2) are clearly satisfied. To verify (3) and (4), note first that each form in $C^\infty(U, \mathcal{G}(U))$ is a sum of forms of the type $a_{i_1\ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. By linearity of d, together with distributivity of exterior multiplication with respect to addition, it suffices to check (3) and (4) on forms of this type.

Property (3). Suppose

$$\mu = a_{i_1\ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \quad\text{and}\quad \tau = b_{j_1\ldots j_\ell}\, dx_{j_1} \wedge \ldots \wedge dx_{j_\ell}.$$

Then

$$\begin{aligned} d(\mu \wedge \tau) &= d[a_{i_1\ldots i_k} b_{j_1\ldots j_\ell}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_\ell}] \\ &= \sum_{r=1}^{n} \left[\frac{\partial}{\partial x_r}(a_{i_1\ldots i_k}) b_{j_1\ldots j_\ell} + a_{i_1\ldots i_k} \frac{\partial}{\partial x_r}(b_{j_1\ldots j_\ell})\right] \\ &\qquad\qquad dx_r \wedge dx_{i_1} \wedge \ldots \wedge dx_{j_\ell} \\ &= \left(\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(a_{i_1\ldots i_k})\, dx_r \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right) \\ &\qquad\qquad \wedge (b_{j_1\ldots j_\ell}\, dx_{j_1} \wedge \ldots \wedge dx_{j_\ell}) \\ &\quad + (-1)^k (a_{i_1\ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}) \\ &\qquad\qquad \wedge \left(\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(b_{j_1\ldots j_\ell})\, dx_r \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_\ell}\right) \\ &= (d\mu) \wedge \tau + (-1)^k \mu \wedge d\tau. \end{aligned}$$

Property (4). For $\mu = a_{i_1\ldots i_k}\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$,

$$\begin{aligned} d^2\mu &= d\left[\sum_{r=1}^{n} \frac{\partial}{\partial x_r}(a_{i_1\ldots i_k})\, dx_r \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right] \\ &= \sum_{r,s=1}^{n} \frac{\partial}{\partial x_s}\left[\frac{\partial}{\partial x_r}(a_{i_1\ldots i_k})\right] dx_s \wedge dx_r \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}. \end{aligned}$$

But certainly the terms in this expression with $r = s$ are zero, since $dx_r \wedge dx_r = 0$. Moreover, for $r \neq s$, the equality of mixed partial derivatives on R^n implies that

$$\frac{\partial}{\partial x_s}\frac{\partial}{\partial x_r}(a_{i_1 \ldots i_k}) = \frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s}(a_{i_1 \ldots i_k}),$$

so that

$$\frac{\partial}{\partial x_s}\frac{\partial}{\partial x_r}(a_{i_1 \ldots i_k})\, dx_s \wedge dx_r = -\frac{\partial}{\partial x_r}\frac{\partial}{\partial x_s}(a_{i_1 \ldots i_k})\, dx_r \wedge dx_s;$$

thus the remaining terms match up in pairs which cancel each other.

Thus the operator d_U has Properties (1)-(4). By uniqueness, every linear operator on $C^\infty(U, \mathcal{G}(U))$ having these properties must be given by the above boxed formula. In particular, if U_1 is any open subset of U, then $\varphi|_{U_1}$ is a coordinate system, and d_{U_1}: $C^\infty(U_1, \mathcal{G}(U_1)) \to C^\infty(U_1, \mathcal{G}(U_1))$ is given in the coordinate system $\varphi|_{U_1}$ by the same formula. Thus, if $\omega \in C^\infty(X, \mathcal{G}(X))$, then

$$d_{U_1}(\omega|_{U_1}) = (d_U(\omega|_U))|_{U_1}.$$

This relation enables us to define d globally by $(d\omega)|_U = d_U(\omega|_U)$ for all

$$\omega \in C^\infty(X, \mathcal{G}(X))$$

and any coordinate neighborhood U. This d is well defined because if U and V are overlapping coordinate neighborhoods, then

$$(d_U(\omega|_U))|_{U \cap V} = d_{U \cap V}(\omega|_{U \cap V}) = (d_V(\omega|_V))|_{U \cap V}.$$

Clearly, d has the required properties, since d_U has them for each U. □

DIGRESSION ON VECTOR ANALYSIS. The multilinear algebra developed above is particularly simple in the case $n = 3$. We want to show how the classical approach of vector analysis fits into the scheme of differential forms.

In order to develop the connection, we consider first the general situation in an n-dimensional vector space T.

Definition. A *volume element* of T is a choice of basis in $\Lambda^n(T^*)$; since $\Lambda^n(T^*)$ is 1-dimensional, a volume element is a choice of a non-zero element in $\Lambda^n(T^*)$.

Example. If T is the tangent space to a manifold and $\{dx_1, \ldots, dx_n\}$ is a basis for T^*, then $dx_1 \wedge \ldots \wedge dx_n$ is a volume element of T. (Note that a volume element ω determines an isomorphism $\Lambda^n(T^*) \cong R^1$, where $r\omega$ corresponds to r. Conversely, such an isomorphism defines a volume element ω corresponding to 1.)

Remark. Given a volume element ω of T, there exists a natural isomorphism m: $\Lambda^{n-1}(T^*) \to T$ defined as follows. Recall that T is naturally isomorphic to its double dual T^{**}. Identifying T^{**} with T through this isomorphism, m will have values in T^{**}. For $\varphi \in \Lambda^{n-1}(T^*)$, $m(\varphi)$ is then defined by $[m(\varphi)](\psi) = \lambda$, where $\psi \in T^*$, λ is the real number such that $\varphi \wedge \psi = \lambda\omega$. To show that m is an isomorphism, let $\{\varphi_1, \ldots, \varphi_n\}$ be a basis for T^* such that $\omega = \varphi_1 \wedge \ldots \wedge \varphi_n$. Then the set $\{\varphi_1 \wedge \ldots \wedge \varphi_{j-1} \wedge \varphi_{j+1} \wedge \ldots \wedge \varphi_n\}$ is a basis for $\Lambda^{n-1}(T^*)$. The value of m on these basis vectors is then given by

$$m(\varphi_1 \wedge \ldots \wedge \varphi_{j-1} \wedge \varphi_{j+1} \wedge \ldots \wedge \varphi_n) = (-1)^{n+j} e_j,$$

where $\{e_1, \ldots, e_n\}$ is the basis for T dual to $\{\varphi_1, \ldots, \varphi_n\}$.

Remark. Given an inner product $\langle\, ,\rangle$ on a finite dimensional vector space T, there exists a natural isomorphism $g\colon T \to T^*$ defined by

$$[g(v)](w) = \langle v, w\rangle \qquad (v, w \in T).$$

If $\{e_1, \ldots, e_n\}$ is a basis for T, let $g_{ij} = \langle e_i, e_j\rangle$, $(i, j \in \{1, \ldots, n\})$. Then in terms of the dual basis $\{\varphi_1, \ldots, \varphi_n\}$ for T^*,

$$g(e_i) = \sum_{j=1}^{n} g_{ij}\varphi_j \qquad (i \in \{1, \ldots, n\}).$$

In particular, if $\{e_1, \ldots, e_n\}$ is orthonormal, then $g_{ij} = \delta_{ij}$, and

$$g(e_i) = \varphi_i.$$

Applications. Take $T = R^n$. Then T has an inner product and a natural volume element $\omega = \varphi_1 \wedge \ldots \wedge \varphi_n$, where $\{\varphi_i\}$ is the dual basis to the natural basis $\{e_i\}$ for R^n. Thus the isomorphisms m and g are defined. Also, we have natural identifications $T(R^n, x) \longleftrightarrow R^n$ for each $x \in R^n$.

1. Let $f \in C^\infty(R^n, R^1)$. Then the *gradient* of f is the vector field on R^n given by

$$\boxed{\operatorname{grad} f = g^{-1} \circ (df).}$$

Relative to the usual coordinates $(x_1, \ldots, x_n) = (r_1, \ldots, r_n)$ on R^n,

$$\operatorname{grad} f = g^{-1} \circ (df) = g^{-1}\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\, dx_j\right) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_j} \longleftrightarrow \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

2. Let V be a vector field on R^3. Then $g \circ V$ is a 1-form and $d(g \circ V)$ is a 2-form. Now for dimension $T = 3$, $\Lambda^2(T^*) = \Lambda^{n-1}(T^*)$, so the isomorphism m maps $\Lambda^2(T^*) \to T$. Thus $m(d(g \circ V))$ is a vector field on R^3. It is called the *curl* of V.

$$\boxed{\operatorname{curl} V = (m \circ d \circ g)(V).}$$

Exercise. Compute the coordinate expression for curl V.

3. Let v_1 and v_2 be vectors in R^3. Then $g(v_1)$ and $g(v_2)$ are 1-forms. Their exterior product is a 2-form; its image under m is a vector, called the *cross product* of v_1 and v_2.

$$\boxed{v_1 \times v_2 = m(g(v_1) \wedge g(v_2)).}$$

4. Let V be a vector field on R^n. Then $m^{-1}(V)$ is an $(n-1)$-form on R^n. Its differential is an n-form, that is, a multiple of the volume element ω. This multiple is (up to sign) the *divergence* of V:

$$\boxed{(-1)^{n-1} d \circ m^{-1}(V) = (\operatorname{div} V)\,\omega.}$$

Remark. Using these formulas, certain important formulas of vector analysis become trivial consequences of $d^2 = 0$.