# LECTURE 10

## Acceleration as Fermi-Walker transport

I. Moving vs. fixed frames
II. Fermi-Walker transport
III. Fermi-Walker transport of a vector

MTW[Ch. 6]

# I. Moving Frame vs. Fixed Frame

The mathematization of accelerative change in the motion of particles/bodies gives rise to parametrized families of inertial reference frames, each one with an observer-defined tetrad of 4-vectors. They form a basis for the 4-d vector space whose origin is on the observer's world line. For an observer subject to uniform linear acceleration this world line is

$$X(\tau):\ \{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\} = \left\{\frac{1}{g}\,\mathrm{sh}\,g\tau,\ \frac{1}{g}\,\mathrm{ch}\,g\tau,\ 0,\ 0\right\},$$

and the corresponding parametrized family of Lorentz orthonormal frames is

$$\begin{aligned}
\mathbb{u}&:\ (\mathbb{u})^\mu \equiv (\mathbb{e}_{0'})^\mu = (\mathrm{ch}\,g\tau,\ \mathrm{sh}\,g\tau,\ 0,\ 0)\\
\frac{\mathbb{a}}{g}&:\ \left(\frac{\mathbb{a}}{g}\right)^\mu \equiv (\mathbb{e}_{1'})^\mu = (\mathrm{sh}\,g\tau,\ \mathrm{ch}\,g\tau,\ 0,\ 0)\\
\mathbb{e}_{2'}&:\ \qquad (\mathbb{e}_{2'})^\mu = (\ 0\ ,\ \ 0\ ,\ 1,\ 0)\\
\mathbb{e}_{3'}&:\ \qquad (\mathbb{e}_{3'})^\mu = (\ 0\ ,\ \ 0\ ,\ 0,\ 1)
\end{aligned} \qquad (10.1)$$

The component of this moving frame, $\{\mathbb{e}_{0'}(\tau), \mathbb{e}_{1'}(\tau), \mathbb{e}_{2'}(\tau), \mathbb{e}_{3'}(\tau)\}$, are typically given relative to a physically fixed basis

$$\begin{aligned}
\mathbb{e}_0&:\ (1, 0, 0, 0)\\
\mathbb{e}_1&:\ (0, 1, 0, 0)\\
\mathbb{e}_2&:\ (0, 0, 1, 0)\\
\mathbb{e}_3&:\ (0, 0, 0, 1).
\end{aligned}$$

Its components are fixed because these basis vectors are aligned with the "fixed" stars.

10.3

## II. Fermi-Walker Transport

The physically essential property of the moving frame, Eq.(10.1), associated with a linear uniformly accelerated observer is that it is non-rotating. Three freely gimbaled gyroscopes aligned along the tree spatial coordinate axes $\bar{x}, \bar{y}, \bar{z}$ in the rocket frame would not precess in the frame of the accelerating rocket as it executed its spacetime trajectory.

Fermi-Walker transport extends this non-rotation feature to a rocket subject to arbitrary acceleration.

### 1.) Fermi-Walker transport: Its construction.

A. GIVEN:

(i) The spacetime trajectory

$$X(\tau): \{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\}$$

of a particle/body subjected to some given acceleration.

(ii) The globally defined inertial LAB frame whose spatial axes $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ point into the direction of some three chosen fixed stars

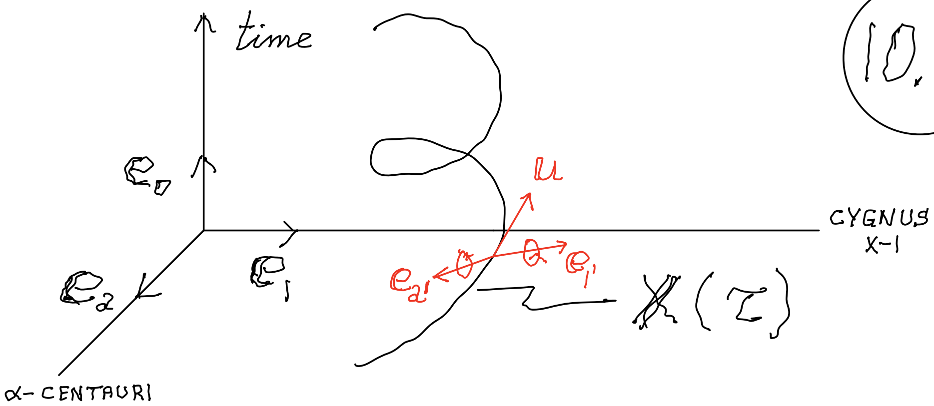


Figure 10.1: *Inertial LAB frame with fixed orientation relative to the FIXED STARS vs an instantaneous Lorentz frame at a point event on $\mathbb{X}(\tau)$.*

(iii) Three GYROSCOPES attached to and carried along $\mathbb{X}(\tau)$, the accelerated world line, but gimbaled freely so that no torques act on each of the gyroscopes.

(iv) The $\tau$-parametrized gyroscope-induced Lorentz orthonormal family of frames

$$\left\{ e_{0'}(\tau) = \frac{d\mathbb{X}(\tau)}{d\tau} \equiv u(\tau),\ e_{1'}(\tau),\ e_{2'}(\tau),\ e_{3'}(\tau) \right\}$$

which consist of

a) the particle's 4-velocity

$$u(\tau) = \frac{d\mathbb{X}}{d\tau} : \left\{ u^\mu(\tau) = \frac{dx^\mu}{d\tau};\ \mu = 0,1,2,3 \right\}$$

and

b) the three orthonormal directions determined by the three respective gyroscope directions.

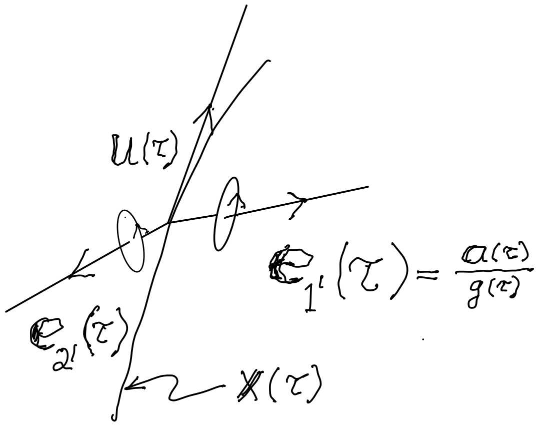


Figure 10.2: Gyroscope and 4-velocity determined Lorentz frame with one of the gyroscope axes aligned along the direction of the local acceleration vector $a$

(v) Notation:

Relative to the global LAB frame the gyroscope-determined 4-vectors have the components

$$e_{1'} : \{ e_{1'}{}^{\mu} : \mu = 0,1,2,3 \}$$
$$e_{2'} : \{ e_{2'}{}^{\mu} : \mu = 0,1,2,3 \}$$
$$e_{3'} : \{ e_{3'}{}^{\mu} : \mu = 0,1,2,3 \}$$
$$u = e_{0'} : \{ u^{\mu} : \mu = 0,1,2,3 \}$$

vs.

relative to the comoving frame these same 4-vectors have the components

$$e_{1'} : \{ e_{1'}{}^{\mu'} : \mu = 0,1,2,3 \} = \{ 0,1,0,0 \}$$
$$e_{2'} : \{ e_{2'}{}^{\mu'} : \mu = 0,1,2,3 \} = \{ 0,0,1,0 \}$$
$$e_{3'} : \{ e_{3'}{}^{\mu'} : \mu = 0,1,2,3 \} = \{ 0,0,0,1 \}$$
$$u = e_{0'} : \{ u^{\mu'} : \mu = 0,1,2,3 \} = \{ 1,0,0,0 \}$$

## B. CONCLUSION: 

(i) The gyroscopes form a spatial platform with an orthonormal basis:

$$\mathbf{e}_{i'} \cdot \mathbf{e}_{j'} \equiv e_{i'}^{\mu'} e_{j'}^{\nu'} \eta_{\mu'\nu'} = \eta_{i'j'} \quad \left( = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \ \begin{matrix} i' \\ j' \end{matrix} \Big\} = 1,2,3 \right)$$

(ii) The time-like basis vector $\mathbf{e}_{0'} \equiv \mathbf{u}$ is Lorentz normalized,

$$\mathbf{e}_{0'} \cdot \mathbf{e}_{0'} = \mathbf{u} \cdot \mathbf{u} = u^{\mu'} u^{\nu'} \eta_{\mu'\nu'} = -1$$

and Lorentz orthogonal to the spatial platform:

$$\mathbf{e}_{0'} \cdot \mathbf{e}_{i'} \equiv \mathbf{u} \cdot \mathbf{e}_{i} = 0 \qquad i' = 1,2,3$$

(iii) Consequently,

$$\mathbf{e}_{\mu'} \cdot \mathbf{e}_{\nu'} = \eta_{\mu'\nu'} \quad \left( = \begin{bmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \right)$$

is a set of 16 frame invariants, and

$$\{\mathbf{e}_{0'}(\tau), \mathbf{e}_{1'}(\tau), \mathbf{e}_{2'}(\tau), \mathbf{e}_{3'}(\tau)\} \equiv B_{FW}$$

is a $\tau$-parametrized family of bases for the $\tau$-parametrized family of instantaneus Lorentz frames (a.k.a. "moving frames").

(iv) Summary:

The $\tau$-parametrized family $B_{FW}$ is determined by the freely gimbaled gyroscopes which are carried along a given accelerated worldline whose tangent is its its instantaneous 4-velocity.

$B_{FW}$ is called a Fermi-Walker basis

## III. Fermi-Walker transport of a vector

Let $v(\tau)$ be a $\tau$-parametrized family of vectors along a given curve $X(\tau)$ in spacetime

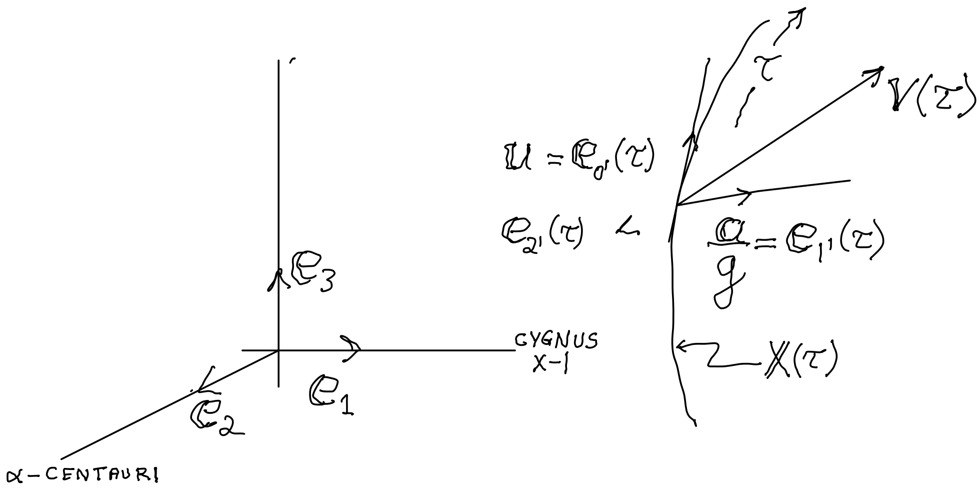


Figure 10.3: Instantaneous Fermi-Walker frame in a global Lab (= "fixed stars") frame.

Expand $v(\tau)$ in terms of the moving Fermi-Walker basis and in terms of the fixed Lab basis. These expansions lead to the following

**Definition** (F-W Transport)

A $\tau$-parametrized family of vectors $v(\tau)$ is said to be Fermi-Walker transported along the given spacetime trajectory $X(\tau)$ if it components remain constant relative to the gyroscope-induced basis

$$v(\tau) = e_{\mu'}(\tau)\underbrace{v^{\mu'}}_{\text{constant}} = \underbrace{e_\mu}_{\text{fixed}\left(\substack{\text{relative to}\\ \text{the fixed stars}}\right)} v^\mu(\tau)$$

Thus, a Fermi-Walker transported family of vectors $V(\tau)$ have components $v^{\mu'}$ which are constant relative to the F-W basis $\{e_{\mu'}(\tau)\}$,

$$\{e_{\mu'}(\tau)\}: \; V(\tau) = e_{\mu'}(\tau)\, v^{\mu'}.$$

10.8

This gyroscope-induced basis changes relative to the fixed stars. On the other hand, $V(\tau)$ has components $v^{\mu}(\tau)$ which are changing relative to the fixed star basis $\{e_{\mu}\}$,

$$\{e_{\mu}\}: \; V(\tau) = e_{\mu}\, v^{\mu}(\tau)$$

## IV. The law governing Fermi-Walker transport

The mathematization of the gyroscope-induced F-W transport consists of setting up the system of differential equations whose solutions are $v^{\mu}(\tau)$, the coefficients of $V(\tau)$ relative to the fixed star basis:

$$e_{\mu}\, v^{\mu}(\tau) = V(\tau) = e_{\mu'}(\tau)\, v^{\mu'}.$$

MTW in their Section 6.5, p 170-171, show that this system of differential equations is linear and is given by

$$\frac{dv^{\mu}}{d\tau} = \left(u^{\mu} a^{\nu} - a^{\mu} u^{\nu}\right) v_{\nu}, \quad \mu = 0,1,2,3$$

or in terms of frame invariant coefficients

$$\frac{dv^{\mu}}{d\tau} = u^{\mu}\, \mathbf{a}\cdot V - a^{\mu}\, \mathbf{u}\cdot V.$$

Here

$$u^{\mu}(\tau) = \frac{dx^{\mu}}{d\tau} \qquad (\text{"4-velocity"})$$

$$a^{\nu}(\tau) = \frac{d^2 x^{\nu}}{d\tau^2} \qquad (\text{"4-acceleration"})$$

$$v_{\nu}(\tau) = \eta_{\nu\sigma}\, v^{\sigma}(\tau)$$

(Nota bene:
In MTW Ex. 6.9 is the problem of solving the F-W system for the case where $\mathbf{v}(\tau)=\mathbf{S}(\tau)$ is the spin 4-vector of a spinning electron in a circular orbit.)

1. Notice that $v^\mu = u^\mu$ is a solution to the above system of equations.
2. Also notice that these equations imply that (i) with $v^\mu = u^\mu$ its rate of change $\frac{dv^\mu}{d\tau} = a^\mu$, and (ii) with $v^\mu = a^\mu$ its rate of change $\frac{dv^\mu}{d\tau} = g^2 u^\mu$.

This means that whatever changes $u^\mu$ and $a^\mu$ are subjected to, that these instantaneous changes are confined to the instantaneous plane spanned by $\mathbf{u}$ and $\mathbf{a}$. There is no rotation out of this plane, i.e. the infinitesimal change in $u^\mu$ and $a^\mu$ is a "pure boost" confined to the instantaneous $\mathbf{u}-\mathbf{a}$ plane.