

LECTURE 13

13.0

I. Addition and Meaning of Covectors

II. Metric on a Vectorspace

III. Natural Isomorphism Between Vectors and their Duals

Chew and assimilate the 2nd Lecture in "The Dual of a Vector Space...".
Then digest the 3rd and 4th Lecture.

I. The geometry of dials in an oblique coordinate system.

(13.1)

Covector as a displacement density.

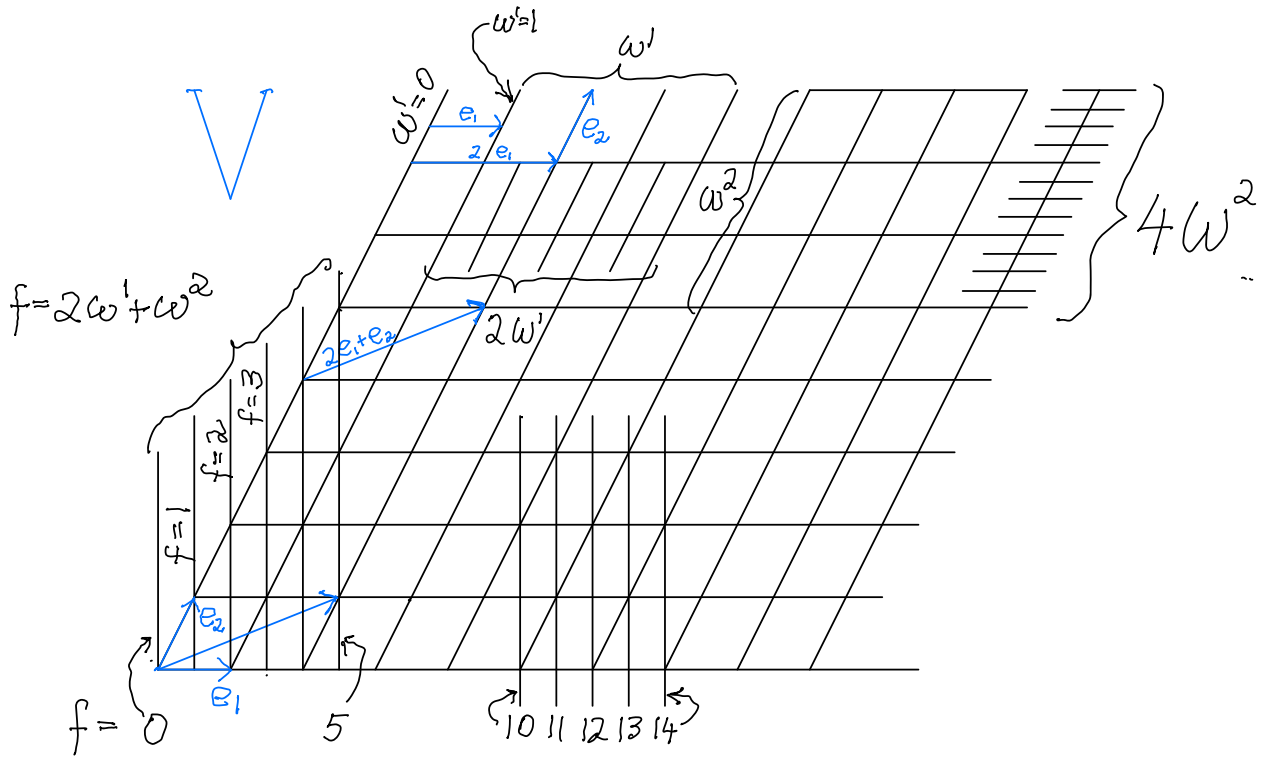


Figure 13.1: Addition of vectors and covectors. With $\{e_1, e_2\}$ as the basis for V , $\{w^1, w^2\}$ is its cobasis, i.e. the basis of V^* .

Given the covector $f = 2w^1 + w^2$, its geometrical properties in the given vector space V are captured by the fact that f ($\equiv \langle f | = f$) refers to the density of its isograms in V . Thus

$$\left. \begin{array}{l} \langle f | e_1 \rangle = 2 \\ \langle f | e_2 \rangle = 1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} \langle f | \frac{e_1}{2} \rangle = 1 \\ \langle f | e_2 \rangle = 1 \end{array} \right\} \Rightarrow \text{The } f=1 \text{ isogram passes through } \frac{e_1}{2} \text{ and } e_2$$

implies that f 's density into the direction of e_1 is 2, i.e. two units (of whatever physical quantity) per displacement standard e_1 , and its density into the direction of e_2 is 1, i.e. one unit per displacement standard e_2 .

The density of $f = \langle f |$ into the direction of the displacement $2e_1 + e_2$ is

$$\begin{aligned} f(2e_1 + e_2) &\equiv \langle f | 2e_1 + e_2 \rangle \\ &= \langle 2\omega^1 + \omega^2 | 2e_1 + e_2 \rangle \\ &= 4 + 0 + 1 + 0 = 5 \quad (\text{"units"}) \end{aligned}$$

which, as shown in Figure 13.1, is the number of integral-valued isograms pierced by the vector $2e_1 + e_2$.

II. Metric on a Vector space: Bilinear Functional.

There is no natural (i.e. unique and basis independent) isomorphism between V and V^* . However, if the vector space has an inner product defined on it, then such an isomorphism is determined.

An inner product is implemented on a vector space by means of the following constellation of concepts.

Definition ("Bilinear Form")

Given: A vector space U and a vector space V .

A bilinear functional (or "form") on $U \times V$ (pairs of elements, one from U and one from V) is a function w ,

$$\begin{aligned} w: U \times V &\longrightarrow \text{reals} \\ (x, y) &\rightsquigarrow w(x, y) \end{aligned}$$

with the properties

$$w(\alpha^1 x_1 + \alpha^2 x_2, y) = \alpha^1 w(x_1, y) + \alpha^2 w(x_2, y)$$

$$w(x, \beta^1 y_1 + \beta^2 y_2) = \beta^1 w(x, y_1) + \beta^2 w(x, y_2).$$

In other words, w is linear in each argument.

Definition ("Metric")

A metric (or inner product) is a bilinear g on $V \times V$ (pairs of elements in V)

$$g: (x, y) \rightsquigarrow g(x, y)$$

with the property $g(x, y) = g(y, x)$.

In other words, a real metric is symmetric in its two arguments

Comment.

1. If the metric were complex-valued, then the symmetry condition gets replaced by

$$g(x, y) = \overline{g(y, x)}$$

2. The metric $g(\cdot, \cdot)$ is called a scalar product whenever g is positive, i.e. $g(x, x) > 0 \forall x \neq 0$.

3. In our development of tensor algebra we shall not insist that g be positive definite. We allow (for physical reasons) for the existence of non-zero vectors such that

$$g(x, x) = 0 \text{ with } x \neq 0.$$

These are "null vectors". We are forced to consider an inner product with such a null result if V is Minkowski spacetime.

Example ("Basis expansion of the metric")

Let

$$x = x^1 e_1 + x^2 e_2 + \dots + x^n e_n$$

be a representation of a vector in terms of a basis $\{e_1, \dots, e_n\}$ for V . Then

$$g(x^1 e_1 + x^2 e_2 + \dots, y^1 e_1 + y^2 e_2 + \dots) = x^1 y^1 g(e_1, e_1) + (x^2 y^1 + x^1 y^2) g(e_1, e_2) + x^2 y^2 g(e_2, e_2) + \dots$$

$$\text{notation} \left\{ \begin{array}{l} \equiv x^1 y^1 e_1 \cdot e_1 + (x^2 y^1 + x^1 y^2) e_1 \cdot e_2 + x^2 y^2 e_2 \cdot e_2 + \dots \\ \equiv x^1 y^1 g_{11} + (x^2 y^1 + x^1 y^2) g_{12} + x^2 y^2 g_{22} + \dots \\ \equiv x^i g_{ij} y^j \equiv [x]^t [g] [y] \end{array} \right.$$

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The coefficients $g_{ij} = e_i \cdot e_j = g(e_i, e_j)$ are the components of the metric g with respect to the basis $\{e_1, e_2, \dots, e_n\}$.

They are the inner products of all pairs of basis vectors.

III. Metric as a Natural Isomorphism Between V and V^*

A metric establishes a natural, i.e. basis independent isomorphism between the vector space V and its space of duals V^* .

To conserve notation, use the same symbol g to designate this correspondence.

A) Its defining property is

$$g: V \rightarrow V^*$$

$$x \rightsquigarrow \underline{x} = g(x) (= "x \cdot ")$$

Here \underline{x} is that linear functional which, when operating on any vector $y \in V$, yields $g(x, y)$:

$$\underline{x} \equiv "x \cdot ": V \rightarrow \mathbb{R}$$

$$y \rightsquigarrow \underline{x}(y) = \underbrace{x \cdot y}_{\langle x | y \rangle} = g(x, y) \quad (\leftarrow \text{"Dirac notation"})$$

B) Representation of g relative to the chosen basis $\{e_i\}$.

One can represent this g relative to any given basis as follows:

Proposition (Basis representation of g)

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Let $\{e_i\}$ be a basis for V

Let $\{\omega^j\}$ be its dual basis for V^* with its property

$$\omega^j(e_i) = \langle \omega^j | e_i \rangle = \delta_{ij} \quad (\text{the "Duality Relation"})$$

Then g is represented relative to the "tensor basis" $\{\omega^i \otimes \omega^j\}$ as the mapping

$$g = g(\cdot, \cdot) = g_{ij} \omega^i \otimes \omega^j : V \times V \longrightarrow \mathbb{R}$$

$$x \rightsquigarrow g(x, \cdot) = g_{ij} \omega^i \otimes \omega^j(x, \cdot) = g_{ij} \langle \omega^i | x \rangle \omega^j$$

$$e_k \rightsquigarrow g(e_k, \cdot) = g_{ij} \langle \omega^i | e_k \rangle \omega^j = g_{kj} \omega^j$$

Comment

1. The tensor product \otimes is the operation which is applied to a pair of linear maps, for example ω^i and ω^j , to obtain the bilinear map

$$\omega^i \otimes \omega^j : V \times V \longrightarrow \mathbb{R}$$

$$(x, y) \rightsquigarrow \omega^i \otimes \omega^j(x, y) = \langle \omega^i | x \rangle \langle \omega^j | y \rangle = x^i y^j$$

2. Use this bilinear map to obtain the components of g :

$$g(e_k, e_l) = g_{ij} \omega^i \otimes \omega^j(e_k, e_l)$$

$$= g_{ij} \langle \omega^i | e_k \rangle \langle \omega^j | e_l \rangle$$

$$= g_{ij} \delta_k^i \delta_l^j$$

$$= g_{kl}$$

More generally, evaluating g on the pair of vectors x and y , one obtains their inner product

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$$\begin{aligned}g_{ij} \omega^i \otimes \omega^j (x, y) &= g_{ij} \langle \omega^i | x \rangle \langle \omega^j | y \rangle \\&= g_{ij} x^i y^j \\&= e_i \cdot e_j x^i y^j \\&= (x^i e_i) \cdot (y^j e_j) \\&= x \cdot y\end{aligned}$$

C) The metric g in its basis representation

$$g = g_{ij} \omega^i \otimes \omega^j \quad (13.1)$$

maps the vector x in its basis representation

$$x = e_k x^k \in V$$

to its image $\underline{x} = x_i \omega^i$ in V^* , where it is represented by

$$\underline{x} = x^k g_{kj} \omega^j \in V^*$$

This mapping process takes direct advantage of the boxed Eq. (13.1) with the result

$$\begin{aligned}g_{ij} \omega^i \otimes \omega^j (e_k x^k, \cdot) &= g_{ij} \underbrace{\langle \omega^i | e_k \rangle}_{\delta_k^i} x^k \omega^j (\cdot) \\&= x^k g_{kj} \omega^j (\cdot) \\&\equiv x^k g_{kj} \langle \omega^j | \quad \in V^*\end{aligned}$$

Thus, the coordinate components x_i of the image of $x = e_k x^k$ produced by g are obtained from x^k by lowering its indices,

$$x_i = x^k g_{kj}.$$

This relation came, of course, from the mapping

depicted in Figure 13.2 below.

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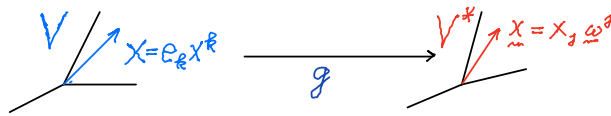


Figure 13.2: The metric g is a mapping between the given vector space V and its dual space V^* .