LECTURE 13

(13.0)

I. Addition and Meaning of Covectors

II. Metric on a Vectorspace

III. Natural Isomorphism Between Vectors and their Duals

Chew and assimilate the 2nd Lecture in "The Dual of a Vector Space...".
Then digest the 3rd and 4th Lecture.

I. The geometry of duals in an oblique coordinate system. (13.1)

Covector as a displacement density.

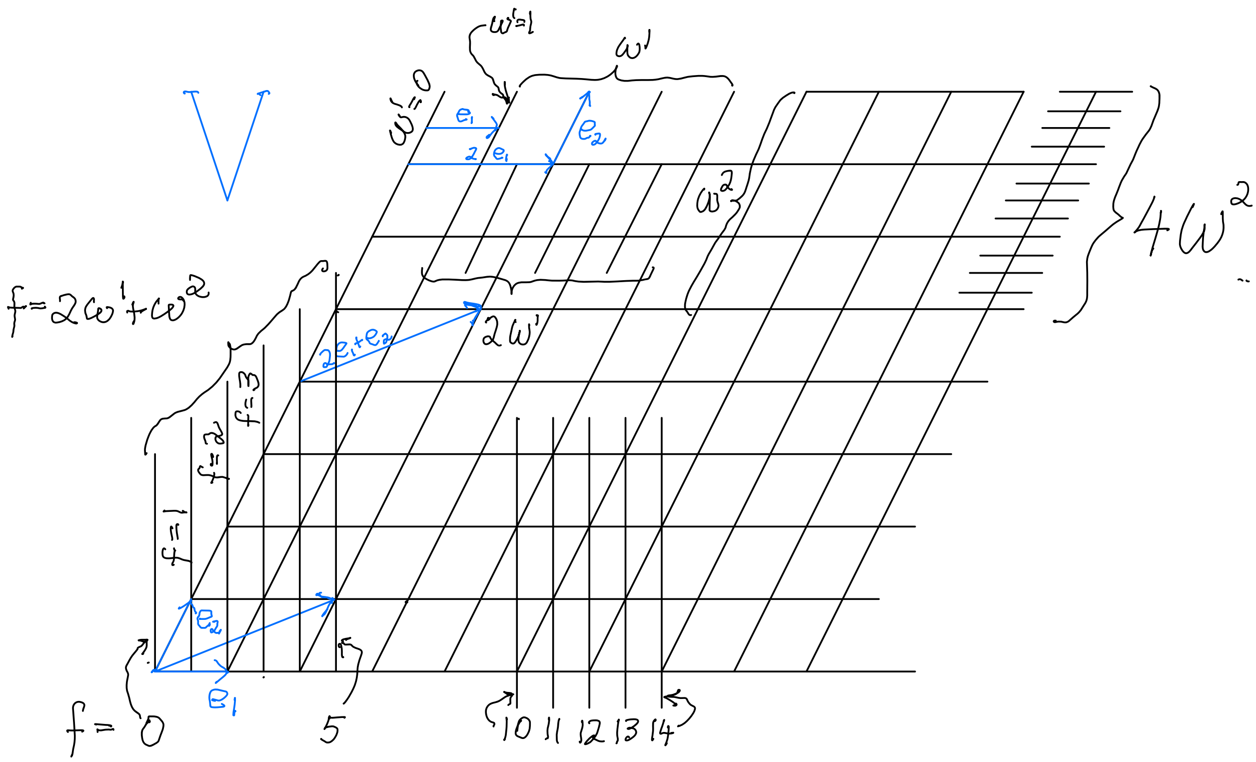

Figure 13.1: Addition of vectors and covectors. With $\{e_1, e_2\}$ as the basis for V, $\{\omega^1, \omega^2\}$ is it cobasis, i.e. the basis of V^*.

Given the covector $f=2\omega^1+\omega^2$, its geometrical properties in the given vector space V are captured by the fact that f ($\equiv \langle f| \equiv \underset{\sim}{f}$) refers to the density of its isograms in V. Thus

$$\left.\begin{array}{l}\langle f|e_1\rangle = 2 \\ \langle f|e_2\rangle = 1\end{array}\right. \left(\begin{array}{l}\Longrightarrow \langle f|\frac{e_1}{2}\rangle = 1 \\ \longrightarrow \langle f|e_2\rangle = 1\end{array}\right\} \Rightarrow \text{The } f=1 \text{ isogram passes through } \tfrac{e_1}{2} \text{ and } e_2 \Big)$$

implies that f's density into the direction of e_1 is 2, i.e. two units (of whatever physical quantity) per displacement standard e_1, and its density into the direction of e_2 is 1, i.e. one unit per displacement standard e_2.

The density of $f = \langle f|$ into the direction of the displacement $2e_1+e_2$ is

$$f(2e_1+e_2) \equiv \langle f|2e_1+e_2\rangle$$
$$\doteq \langle 2\omega^1+\omega^2|2e_1+e_2\rangle$$
$$= 4+0+1+0 = 5 \quad (\text{"units"})$$

which, as shown in Figure 13.1, is the number of integral-valued isograms pierced by the vector $2e_1+e_2$.

II. Metric on a Vector space: Bilinear Functional

There is no natural (i.e. unique and basis independent) isomorphism between V and V^*. However, if the vector space has an inner product defined on it, then such an isomorphism is determined.

An inner product is implemented on a vector space by means of the following constellation of concepts.

Definition ("Bilinear Form")

Given: A vector space U and a vector space V.

A <u>bilinear functional</u> (or "form") on $U\times V$ (pairs of elements, one from U and one from V) is a function w,

$$w: \quad U\times V \longrightarrow \text{reals}$$
$$(x,y) \rightsquigarrow w(x,y)$$

with the properties

$$w(\alpha^1 x_1+\alpha^2 x_2, y) = \alpha^1 w(x_1,y)+\alpha^2 w(x_2,y)$$
$$w(x,\beta^1 y_1+\beta^2 y_2) = \beta^1 w(x,y_1)+\beta^2 w(x,y_2).$$

In other words, w is <u>linear</u> in each argument.

Definition ("Metric")

A metric (or inner product) is a bilinear g on $V \times V$ (pairs of elements in V)

$$g: \ (x,y) \rightsquigarrow g(x,y)$$

with the property $g(x,y) = g(y,x)$.

In other words, a real metric is <u>symmetric</u> in its two arguments

Comment.

1. If the metric were complex-valued, then the symmetry condition gets replaced by

$$g(x,y) = \overline{g(y,x)}$$

2. The metric $g(\ ,\ )$ is called a <u>scalar product</u> wherever g is positive, i.e. $g(x,x)>0 \ \forall x \neq 0$.
3. In our development of tensor algebra we shall not insist that g be positive definite. We allow (for physical reasons) for the existence of non-zero vectors such that

$$g(x,x) = 0 \quad \text{with} \quad x \neq 0.$$

These are "null vectors. We are forced to consider an inner product with such a null result if V is Minkowski spacetime.

Example ("Basis expansion of the metric")

Let

$$x = x^1 e_1 + x^2 e_2 + \cdots + x^n e_n$$

be a representation of a vector in terms of a basis $\{e_1, \cdots, e_n\}$ for V. Then

$$g(x^1e_1 + x^2e_2 + \cdots, y^1e_1 + y^2e_2 + \cdots) = x^1y^1 g(e_1,e_1) + (x^2y^1 + x^1y^2) g(e_1,e_2) + x^2y^2 g(e_2,e_2) + \cdots$$

$$\text{notation} \left\{ \begin{aligned} &\equiv x^1y^1\, e_1\cdot e_1 + (x^2y^1 + x^1y^2)\, e_1\cdot e_2 + x^2y^2\, e_2\cdot e_2 + \cdots \\ &\equiv x^1y^1\, g_{11} + (x^2y^1 + x^1y^2)\, g_{12} + x^2y^2\, g_{22} + \\ &\equiv x^i g_{ij} y^j \equiv [x]^t [g][y] \end{aligned} \right.$$

The coefficients $g_{ij} = e_i \cdot e_j = g(e_i, e_j)$ are the <u>components of the metric g with respect to the basis $\{e_1, e_2, \cdots, e_n\}$.</u>
They are the inner products of all pairs of basis vectors.

III. Metric as a Natural Isomorphism Between V and V^*

A metric establishes a <u>natural</u>, i.e. basis independent isomorphism between the vector space V and its space of duals V^*.
To conserve notation, use the same symbol g to designate this correspondence.

A) It defining property is

$$\boxed{\begin{aligned} g:\ & V \longrightarrow V^* \\ & x \rightsquigarrow \underset{\sim}{x} = g(x, \ ) \ (= \text{"}x\cdot\ \text{"}) \end{aligned}}$$

Here $\underset{\sim}{x}$ is that linear functional which, when operating on any vector $y \in V$, yields $g(x,y)$:

$$\underset{\sim}{x} \equiv \text{"}x\cdot\text{"} : V \longrightarrow R$$

$$y \rightsquigarrow \underbrace{\underset{\sim}{x}(y)}_{\langle x | y \rangle} = x \cdot y = g(x, y)$$

($\longleftarrow$"Dirac notation")

B) Representation of g relative to the chosen basis $\{e_i\}$.

One can represent this g relative to any given basis as follows:

Proposition (Basis representation of g)

13.5

Let $\{e_i\}$ be a basis for V

Let $\{\omega^j\}$ be its dual basis for V^* with its property

$\omega^j(e_i) \equiv \langle \omega^j | e_i \rangle = \delta^j_i$ (the "Duality Relation")

Then g is represented relative to the "tensor basis" $\{\omega^i \otimes \omega^j\}$ as the mapping

$$\boxed{\begin{aligned} g = g(\ ,\ ) = g_{ij}\, \omega^i \otimes \omega^j :\ & V \longrightarrow V^* \\ & x \rightsquigarrow g(x,\ ) = g_{ij}\, \omega^i \otimes \omega^j(x,\ ) = g_{ij} \langle \omega^i | x \rangle \omega^j \end{aligned}}$$

$$e_k \rightsquigarrow g(e_k,\ ) = g_{ij} \langle \omega^i | e_k \rangle \omega^j = g_{kj}\, \omega^j$$

Comment

1. The tensor product $\otimes$ is the operation which is applied to a pair of linear maps, for example ω^i and ω^j, to obtain the bilinear map

$$\omega^i \otimes \omega^j : V \times V \longrightarrow R$$

$$(x, y) \rightsquigarrow \omega^i \otimes \omega^j (x, y) = \langle \omega^i | x \rangle \langle \omega^j | y \rangle$$
$$= x^i y^j$$

2. Use this bilinear map to obtain the components of g:

$$\begin{aligned} g(e_k, e_\ell) &= g_{ij}\, \omega^i \otimes \omega^j (e_k, e_\ell) \\ &= g_{ij} \langle \omega^i | e_k \rangle \langle \omega^j | e_\ell \rangle \\ &= g_{ij}\, \delta^i_k\, \delta^j_\ell \\ &= g_{k\ell} \end{aligned}$$

More generally, evaluating g on the pair of vectors x and y, one obtain their inner product

$$g_{ij}\, \omega^i \otimes \omega^j (x, y) = g_{ij} \langle \omega^i | x \rangle \langle \omega^j | y \rangle$$
$$= g_{ij}\, x^i\, y^j$$
$$= e_i \cdot e_j\, x^i y^j$$
$$= (x^i e_i) \cdot (y^j e_j)$$
$$= x \cdot y$$

C) The metric g in its basis representation

$$\boxed{g = g_{ij}\, \omega^i \otimes \omega^j} \qquad (13.1)$$

maps the vector x in its basis representation

$$x = e_k x^k \in V$$

to its image $\underset{\sim}{x} = x_j \omega^j$ in V^*, where it is represented by

$$\underset{\sim}{x} = x^k g_{kj}\, \omega^j \qquad \in V^*$$

This mapping process takes direct advantage of the boxed Eq. (13.1) with the result

$$g_{ij}\, \omega^i \otimes \omega^j (e_k x^k, \cdot) = g_{ij} \underbrace{\langle \omega^i | e_k \rangle}_{\delta^i_{\ k}} x^k \omega^j(\cdot)$$
$$= x^k g_{kj}\, \omega^j(\cdot)$$
$$\equiv x^k g_{kj} \langle \omega^j | \qquad \in V^*$$

Thus, the coordinate components x_j of the image of $x = e_k x^k$ produced by g are obtained from x^k by <u>lowering</u> its indeces,

$$x_j = x^k g_{kj}\,.$$

This relation came, of course, from the mapping depicted in Figure 13.2 below.

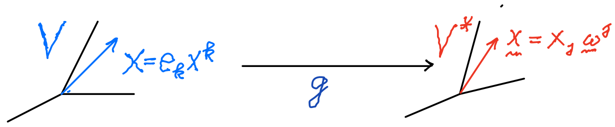

Figure 13.2: The metric g is a mapping betwen the given vector space V and its dual space V^*.