# LECTURE 14



I. The inverse of $g: V \longrightarrow V^*$

II. Reciprocal Basis

III. The preimage of a covector

IV. The inverse metric

V. Summary questions

Read "The Dual of a Vector Space

I. A key point of Lecture 13 is that the metric $g$ maps vectors to covectors, i.e. linear functions on $V$, and that if one chooses a particular basis $\{e_i\}$ and its dual $\{\omega^j\}$, then

$$g = g_{ij}\,\omega^i \otimes \omega^j \quad \left(= \omega^i \otimes g_{ij}\,\omega^j\right)$$

and this mapping, Figure 13.2,

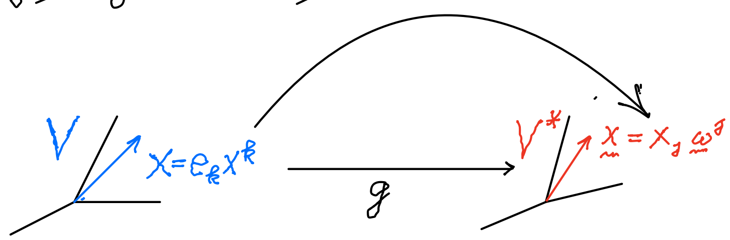


Figure 13.2: The metric $g$ is a mapping between the given vector space $V$ and its dual space $V^*$.

is concretized by the statement

$$\{X^k\} \rightsquigarrow \{X_j = g_{jk} X^k\}.$$

We know already that $\dim V = \dim V^* (=n)$ and that consequently $g$ is one-to-one and that its inverse exists. But then the questions are:

(i) geometrically, what is the vector $g^{-1}(\underset{\sim}{X})$, the inverse image of some covector $\underset{\sim}{X} = \omega^j X_j$ ?

(ii) algebraically, how does one calculate the components of that vector?

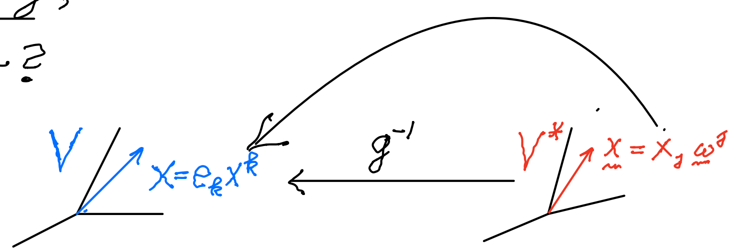


Figure 14.1: The "inverse metric" $g^{-1}$ is the mapping from the dual space $V^*$ back to the given vector space $V$.

The most direct answer to both of these questions is in terms of the basis reciprocal to the one chosen, $\{e_i\}$.

# II. Reciprocal basis: Geometrical origin

Consider the three-dimensional vector space $V = R^3$ with an oblique basis $\{e_1, e_2, e_3\}$.

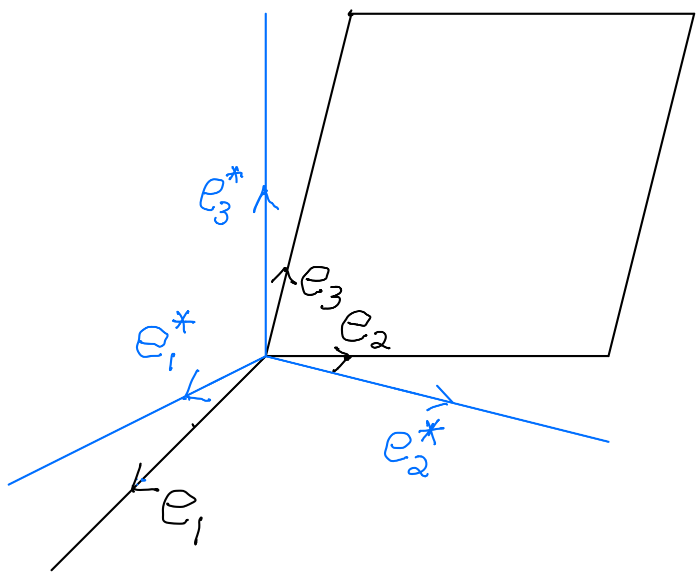


Figure 14.2: An oblique basis $\{e_1, e_2, e_3\}$ on a vector space with an inner product has a corresponding oblique reciprocal basis $\{e_1^*, e_2^*, e_3^*\}$. $e_1^*$ is perpendicular to the plane spanned by $e_2$ and $e_3$.

The vector space $R^3$ with a coordinate basis $\{e_1, e_2, e_3\}$ has three coordinate planes.

The 1-2 plane is spanned by $e_1, e_2$

The 2-3 plane " " " $e_2, e_3$

The 3-1 plane " " " $e_3, e_1$

The vectors perpendicular to these planes form the reciprocal basis

$$\{e_3^*, e_1^*, e_2^*\}.$$

They are mathematized by the reciprocity requirement $e_k^* \cdot e_i = \delta_{ki} = \begin{cases} 0 & k \neq i \\ 1 & k = i \end{cases}$.

Comment

The reciprocal of this requirement, $e_i \cdot e_k^* = \delta_{ik}$, leads to the reciprocal conclusion: The vectors perpendicular to

The $1^*$-$2^*$ plane which is spanned by $e_1^*, e_2^*$ has $e_3$ perpendicular to it.
The $2^*$-$3^*$ plane " " " " $e_2^*, e_3^*$ has $e_1$ perpendicular to it.
The $3^*$-$1^*$ plane " " " " $e_3^*, e_1^*$ has $e_2$ perpendicular to it.

These lead back to the given basis

$$\{e_1, e_2, e_3\}.$$

These geometrical observations are condensed into the following

Definition: ("reciprocal basis")

Given: (i) $g = $ "$\cdot$", a metric on $V$.

(ii) $\{e_i\}$, a basis for $V$.

Then the set of vectors

$$\{e_1^*, e_2^*, \cdots, e_n^*\}$$

with the reciprocity property –

$$e_k^* \cdot e_i = \delta_{ki}$$

is the basis <u>reciprocal</u> to $\{e_i\}$.

Comment:

1. $e_k^*$ is the unique vector which

   (a) is <u>perpendicular</u> to the $(n-1)$-dimensional plane spanned by

   $$\{e_1, \cdots, e_{k-1}, e_{k+1}, \cdots, e_n\},$$

   i.e.

   $$\boxed{e_k^* \cdot e_i = 0} \quad i \neq k$$

   (b) is <u>scaled</u> such that

   $$\boxed{e_k^* \cdot e_k = 1} \quad \text{no sum}$$

   determine $e_k^*$ uniquely

2. $\{e_k^*\}$ is reciprocal to $\{e_i\}$ and $\{e_i\}$ is reciprocal to $\{e_k^*\}$.

3. The projection onto the $k^{th}$ reciprocal basis vector $e_k^*$ yields the $k^{th}$ coordinate $x^k$ of the vector $x = x^i e_i$:

$$x \cdot e_k^* = (x^i e_i) \cdot e_k^* = x^i \delta_{ik}$$

$$\boxed{x \cdot e_k^* = x^k} \qquad (14.1)$$

III. The key to mathematizing the inverse metric $g^{-1}$ is to focus attention on the isograms (level surfaces) of any given $\underset{\sim}{a} \in V^*$. An isogram of $\underset{\sim}{a}$ is the set of points $x$ in $V$ where the linear function has constant value, say, $\underset{\sim}{a}(x) = a_0$:

$$\{x \in V : \underset{\sim}{a}(x) = a_0\} \equiv V_{a_0} \subset V$$

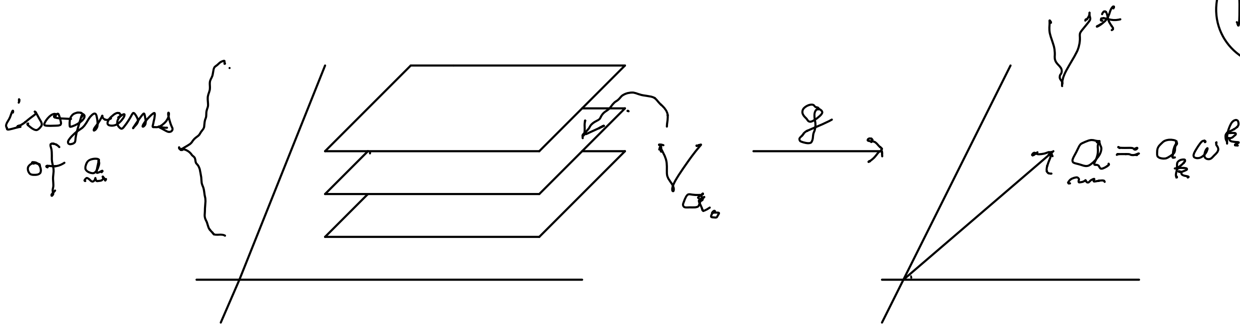


Figure 14.3: Covector $\underset{\sim}{a}$ and its isograms in $V$.

Let $x$ be an arbitrary vector in $V$. Evaluating $\underset{\sim}{a}$ on this vector yields

$$\langle \underset{\sim}{a} | x \rangle = a_k \langle \omega^k | x \rangle$$
$$= a_k x^k .$$

In light of $x \cdot e_k^* = x^k$, Eq. (14.1) on page 14.5, this becomes

$$\langle \underset{\sim}{a} | x \rangle = a_k e_k^* \cdot x$$

or

$$\langle \underset{\sim}{a} | x \rangle = \vec{a} \cdot \vec{x} .$$

$$\boxed{\langle \underset{\sim}{a} | x \rangle \equiv \vec{a} \cdot \vec{x} .} \qquad (14.2)$$

Consider the $\underset{\sim}{a}$ isogram $\{\vec{x} : \underset{\sim}{a}(x) \equiv \langle \underset{\sim}{a} | x \rangle = 0\}$, namely the locus of points where

$$0 = \vec{a} \cdot x$$

Thus, given the covector $\underset{\sim}{a} = a_k \omega^k$ in $V^*$, the corresponding vector $\vec{a}$ in $V$

(i) is perpendicular to all the (parallel) isograms of $\underset{\sim}{a}$, and
(ii) is given by

$$\vec{a} = a_k e_k^* . \qquad (14.3)$$

Consequently, the inverse image of $\underset{\sim}{a} = a_k \omega^k$,

$$g^{-1}(\underset{\sim}{a}) = a_k e_k^* ,$$

is the unique vector with the geometrical property that is perpendicular to the isograms of $\underset{\sim}{a}$

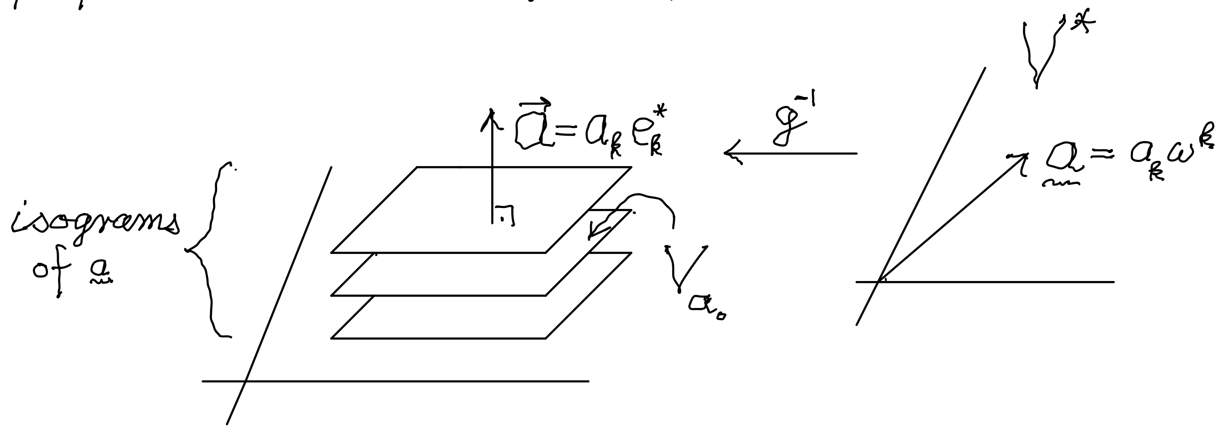


Figure 14.4: The preimage of the covector $\underset{\sim}{a}=a_k\omega^k$, $g^{-1}(\underset{\sim}{a})$, is the unique vector

$$\vec{a}=g^{-1}(a_k\omega^k)=a_k e_k^*$$

which is perpendicular to the isograms of $\underset{\sim}{a}$.

The $\{e_i\}$ basis components of the preimage vector $\vec{a}$, Eq. (14.3), on page 14.6 are a consequence of the matrix $[g^{j\ell}]$ which is the inverse of $[g_{\ell i}]$.

The to-be-dermined representation

$$\vec{a}=e_k a^k$$

for the $a^k$ expansion coefficients is based on the following calculation: In $\langle \underset{\sim}{a}|x\rangle \equiv \vec{a}\cdot\vec{x}$, Eq. (14.2) on page 14.6, let $x=e_\ell$. One obtains

$$\vec{a}\cdot e_\ell=\langle \underset{\sim}{a}|e_\ell\rangle$$

$$a^k e_k\cdot e_\ell=\langle a_j\omega^j|e_\ell\rangle$$

$$a^k g_{k\ell}=a_j\delta^j_\ell=a_\ell$$

Taking advantage of the fact that the $g^{\ell i}$ are the matrix elements of the inverse $g^{-1}$ of $g$, namely that

$$g_{k\ell}\, g^{\ell i} = \delta_k^{\;i}, \quad (\Longleftrightarrow g g^{-1} = I)$$

one finds

$$a^i = a_k g^{ki}$$

and hence,

$$\vec{a} = a^i e_i = a_k g^{ki} e_i .$$

IV. These calculations are summarized by the following

Definition ("inverse metric")

Given: (i) The basis $\{e_i\}$ for the vector space $V$, and $\{\omega^j\}$ for $V^*$

(ii) The metric whose matrix components relative to this basis are

$$e_k \cdot e_\ell = g_{k\ell}$$

(iii) The inverse of this matrix: $g_{k\ell}\, g^{\ell i} = \delta_k^{\;i}$

Then the inverse metric is

$$\begin{aligned} g^{-1}: \quad & V^* \longrightarrow V \\ & \underset{\sim}{a} = a_\ell \omega^\ell \rightsquigarrow \vec{a} = g^{-1}(\underset{\sim}{a}) = a_\ell g^{\ell i} e_i \end{aligned} \qquad (14.4)$$

V. Questions: True or False and if so why?

1. $e_\ell \overset{g}{\rightsquigarrow} g_{\ell j}\, \omega^j$

2. $\vec{a}: \{a^\ell\} \overset{g}{\rightsquigarrow} \underset{\sim}{a}: \{a_j = a^\ell g_{\ell j}\}$

3. $\omega^j \overset{g^{-1}}{\rightsquigarrow} g^{j\ell} e_\ell$

4. $\underset{\sim}{a}: \{a_i\} \overset{g^{-1}}{\rightsquigarrow} \vec{a}: \{a^{kj} a_i\}$

5. $e_k^* \cdot e_\ell^* = g_{k\ell}$

6. $e_i = g_{i\ell}\, e_\ell^*$

7. $e_k^* = g^{k\ell} e_\ell$