LECTURE 17

17.1

I. Flux Vector

II. Flux Tensor

III. Generalization to 4-d spacetime

(17.1b)

Aside from Newton's theory of gravitation, the historically most far reaching theory is the Atomic Theory of matter. It featured a fierce intellectual battle during the 18th and 19th centuries. Here is how David Harriman describes it in his

"THE LOGICAL LEAP: INDUCTION IN PHYSICS".

Scientists need objective standards for evaluating theories. Nowhere is this need more apparent than in the strange history of the atomic theory of matter. Prior to the 19th century, there was little evidence for the theory — yet many natural philosophers believed that matter was made of atoms, and some even wasted their time constructing imaginative stories about the nature of the fundamental particles. Then, during the 19th century, a bizarre reversal occurred: As strong evidence for the theory accumulated rapidly, many scientist rejected the idea of atoms and even crusaded against it.

Both of these errors — the dogmatic belief that was unsupported by evidence, followed by the dogmatic scepticism that ignored abundant evidence — where based on false theories of knowledge. The atomists of ancient Greece were rationalists, i.e., they believed that knowledge can be acquired by reason alone, independent of sensory data. The 19th century skeptics were modern empiricists, i.e., they believed that knowledge
is merely a description of sensory data and therefore references to unobservable entities are meaningless. But scientific

17.1c

knowledge is not the floating abstractions of rationalists or the perceptual descriptions of empiricists; it is the grasp of causal relationships identified by means of the inductive method.

The ensuing Lecture 17 introduces the mathematical method which integrates the atomic theory and the continuum theory into an organic whole. In three-dimesional space this is the particle flux 2-form. In four-dimensional spacetime this is the particle density-flux 3-form.

17.2

The process of mathematizing experiments and observations of the physical world are concretized among others by formulating the flow of matter in term of the Levi-Civita tensor. The result of this formulation is the <u>flux tensor</u>.

I. The Flux Vector

Consider an element of moving fluid consisting particles

(i) having uniform velocity

$$\vec{v} = v^i e_i = \left[\frac{(\text{displacement})}{(\text{time})} \right],$$

uniform particle density

$$N = \left[\frac{(\text{number})}{(\text{volume})} \right],$$

and passing through some chosen area spanned by $\vec{A}_1$ and $\vec{A}_2$ and hence

(ii) occupying during time Δt the interior of the parallelogram spanned by

$$\vec{A}_1,\ \vec{A}_2, \text{ and } \vec{v}\Delta t.$$

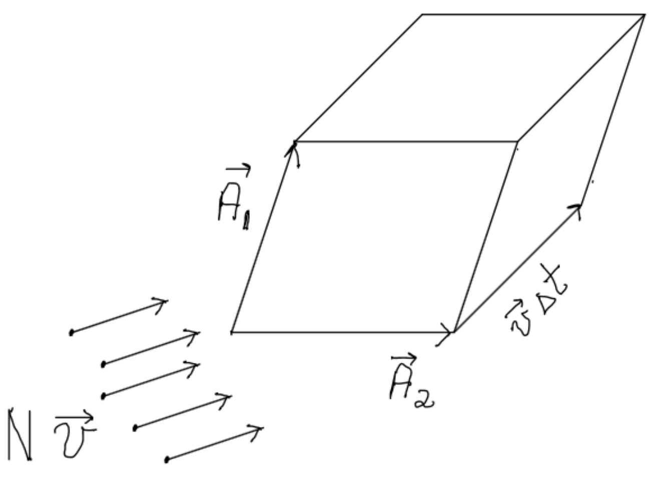

Figure 17.1: Parallelogram spanned by $\vec{A}_1, \vec{A}_2, \vec{v}\Delta t$ during time Δt.

17.3

The particle current, i.e., the measured number of particles (crossing the area $\vec{A}_1\times\vec{A}_2$) per unit time Δt, is $N\,\vec{v}\cdot\vec{A}_1\times\vec{A}_2$. In particular

$$\frac{1}{\Delta t}\overbrace{\underbrace{N(\vec{v}}_{\text{"particle flux"}}\Delta t)\cdot\underbrace{\vec{A}_1\times\vec{A}_2}_{\text{"area"}}}^{\text{"\# of particles in volume } (\vec{v}\Delta t)\cdot\vec{A}_1\times\vec{A}_2\text{"}} = \underbrace{\left[\frac{(\text{particles})}{(\text{area})(\text{time})}\right]}_{\text{"particle flux"}}(\text{area}) \qquad (17.1)$$

This construction on the l.h.s. of Eq. (17.1) is the <u>particle current</u>

$$\boxed{\frac{d(\#)}{dt} = N\,\vec{v}\cdot\vec{A}_1\times\vec{A}_2}\ , \qquad (17.2)$$

the number of particles per unit time crossing the area spanned by $\vec{A}_1$ and $\vec{A}_2$. The product of the particle density N and the common velocity $\vec{v}$ is the <u>particle flux vector</u>,

$$\boxed{N\vec{v} \equiv \mathbf{j} = \text{"particle flux vector"}} \qquad (17.3)$$

II. The Flux Tensor

The above particle current system consists of

(i) the element of fluid with its velocity $\vec{v}$ and density N, which are the given properties of the fluid, and of

(ii) the area $\vec{A}_1\times\vec{A}_2$, which is chosen.

Consequently, one must view the above expression (17.1) as a linear mapping of $V\times V$ into R. This observation leads to the definition of the <u>particle flux</u> $*\mathbf{j}$:

$$*\mathbf{j}:\ (\vec{A}_1,\vec{A}_2) \rightsquigarrow *\mathbf{j}(\vec{A}_1,\vec{A}_2) = N\,\vec{v}\cdot\vec{A}_1\times\vec{A}_2 \ \left(=\frac{d(\#)}{dt}\right) \qquad (17.4)$$

Comment

17.4

The definition of $*j$ in Eq.(17.4) is deficient. This is because the mathematization of the number of particles in an element of fluid in terms of a vector crossproduct is restricted to 3-dimensional space. It does not generalize to 4-dimensional spacetime. However, its representation in terms of the corresponding determinant,

$$N\vec{v}\cdot\vec{A}_1\times\vec{A}_2 = N\det\begin{vmatrix} v^1 & v^2 & v^3 \\ A_1^1 & A_1^2 & A_1^3 \\ A_2^1 & A_2^2 & A_2^3 \end{vmatrix} \qquad (17.5)$$

$$= N\det\begin{vmatrix} v^1 & v^2 & v^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix} \qquad (17.6)$$

does not suffer from this deficiency. Indeed, the following line of reasoning in three dimensions is readily extended to four dimensions.

(i) Apply the expansion of the determinant in Eqs(17.5)-(17.6) in terms of tensor products to Eq.(17.4) on page 17.3 and obtain

$$\begin{aligned} *j(\vec{A}_1,\vec{A}_2) = \{ & N v^1(\omega^2\otimes\omega^3-\omega^3\otimes\omega^2) \\ & + N v^2(\omega^3\otimes\omega^1-\omega^1\otimes\omega^3) \\ & + N v^3(\omega^1\otimes\omega^2-\omega^2\otimes\omega^1)\}(\vec{A}_1,\vec{A}_2) \end{aligned}$$

(ii) Introduce the antisymmetric wedge product $\omega^2\wedge\omega^3 \equiv \omega^2\otimes\omega^3-\omega^3\otimes\omega^2$; more generally,

$$\omega^j\otimes\omega^k-\omega^k\otimes\omega^j \equiv \omega^j\wedge\omega^k = -\omega^k\wedge\omega^j$$

and the totally antisymmetric Levi-Civita symbol in three dimensions

$$\epsilon_{ijk} = \begin{cases} 0 & \text{whenever any two indeces coincide} \\ \epsilon_{123} & \text{whenever } (ijk) \text{ is an } \underline{\text{even}} \text{ permutation of } (123) \\ -\epsilon_{123} & \text{whenever } (ijk) \text{ is an } \underline{\text{odd}} \text{ permutation of } (123)\,. \end{cases}$$

This leads to

$$*\mathbf{j}(\vec{A}_1,\vec{A}_2)=\Big\{\frac{N}{2}\Big[\; v^1\epsilon_{123}\,\omega^2\wedge\omega^3+v^2\epsilon_{231}\,\omega^3\wedge\omega^1+v^3\epsilon_{312}\,\omega^1\wedge\omega^2+ \qquad (17.5)$$
$$+v^1\epsilon_{132}\,\omega^3\wedge\omega^2+v^2\epsilon_{213}\,\omega^1\wedge\omega^3+v^3\epsilon_{321}\,\omega^2\wedge\omega^1\Big]\Big\}\,(\vec{A}_1,\vec{A}_2)$$
$$= N\,v^i\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!\;\;(\vec{A}_1,\vec{A}_2), \qquad (17.7)$$

which holds for all $(\vec{A}_1,\vec{A}_2)\in V\times V$.

(iii) Suppressing explicit reference to its arguments, the particle flux is

$$\boxed{*\mathbf{j}=N\,v^i\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!} \quad \text{"particle flux 2-form"} \qquad (17.8)$$

Comments

1.) The factor $\frac{1}{2!}$ arises because both $\epsilon_{i23}\,\omega^2\wedge\omega^3$ and $\epsilon_{i32}\,\omega^3\wedge\omega^2$ and other equal terms like it are in the implied double sum $\sum_k\sum_\ell$. Thus the double sum $\epsilon_{ijk}\,\omega^j\wedge\omega^k$ $=\epsilon_{ijk}\,\omega^j\otimes\omega^k-\epsilon_{ijk}\,\omega^k\otimes\omega^j$ on the r.h.s. of Eq. (17.7) has each non-zero term repeated twice. The factor $\frac{1}{2!}$ prevents the double sum $\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!$ from referring to an excessive number of tensor products.

Because of this, MTW and others often introduce the <u>restricted</u> double sum without the factor $\frac{1}{2}$:

$$\boxed{*\mathbf{j}=N\,v^i\epsilon_{i|jk|}\,\omega^j\wedge\omega^k\equiv N\,v^i\sum_{j<k}\sum\epsilon_{ijk}\,\omega^j\wedge\omega^k}$$

2.) Physically the expression $*\mathbf{j}$ is the number of particles per unit time passing through the (oriented) area spanned by a pair of as-yet-unspecified vectors

$$*\mathbf{j}=\text{"}\left[\frac{(\text{particles})}{(\text{time})(\text{area})}\right]\text{"}$$

17.6

3.) Mathematically one says that $*\mathbf{j}$, an antisymmetric rank$\binom{0}{2}$ tensor,

$$*\mathbf{j}: \quad V\times V \longrightarrow R$$
$$(\vec{A}_1,\vec{A}_2) \rightsquigarrow *\mathbf{j}(A_1,A_2) = N v^i \epsilon_{i|jk|}\, \omega^j\wedge\omega^k(\vec{A}_1,\vec{A}_2),$$

which is a scalar-valued two-form.

4.) The fact that Eq. (17.7) is a (basis independent) scalar implies that this scalar is an inner product of $\mathbf{j} = N v^i e_i$, the particle flux vector, and another vector which is entirely independent of N and v^i. This vector is identified with the help of the inner products $g_{m\ell} = e_m\cdot e_\ell$ and their inverse matrix $g^{\ell i}$;

$$e_m\cdot e_\ell\, g^{\ell i} = \delta_m{}^i .$$

Introduce this unit matrix into Eq.(17.7)

$$*\mathbf{j}(\vec{A}_1,\vec{A}_2) = N v^m e_m\cdot e_\ell\, g^{\ell i}\,\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!\,(\vec{A}_1,\vec{A}_2)$$
$$= \mathbf{j}\cdot \underbrace{e_\ell\, g^{\ell i}\,\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!}_{\equiv\; e_\ell d^2\Sigma^\ell}\,(\vec{A}_1,\vec{A}_2)$$

and infer that the to-be-intified vector is

$$e_\ell\otimes d^2\Sigma^\ell(\vec{A}_1,\vec{A}_2) = e_\ell\otimes g^{\ell i}\,\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!\,(\vec{A}_1,\vec{A}_2).$$

This vector is clearly orthogonal to both $\vec{A}_1$ and $\vec{A}_2$. In fact, relative to an orthonormal frame it is the familiar cross product

$$\boxed{\vec{A}_1\times\vec{A}_2 = e_\ell \otimes d^2\Sigma^\ell(A_1,A_2).}$$ "cross product"

III. Generalizations to 4-d spacetime

1.) The generalization of the 3-d vector cross product to 4-d spacetime is

$$\boxed{e_\sigma\otimes d^3\Sigma^\sigma(A_1,A_2,A_3) = e_\sigma\otimes g^{\sigma\mu}\epsilon_{\mu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma/3!\,(A_1,A_2,A_3)}\,.$$

17.7

2.) The generalization of the 3-d "particle flux 2-form"

$$\boxed{{}^*\mathbf{J} = N v^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2!}$$

to the 4-d "particle density-flux 3-form" is

$$\boxed{{}^*\mathbb{J} = N u^\mu \epsilon_{\mu\alpha\beta\gamma}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma / 3!}$$

3.) The generalization of the 3-d "particle flux vector"

$$\boxed{\mathbf{J} = N \vec{v} = N v^i e_i}$$

to the 4-d "particle density flux 4-vector" is

$$\boxed{\mathbb{J} = N\mathbf{u} = N u^\mu e_\mu}$$

where $\mathbf{u}$ is the common particle 4-velocity