LECTURE 18

18.1

I. Flux tube structures

a) Magnetic flux density

b) Flux density of electric force lines

c) Flux density of lines of particle motion

In MTW Sections 4.2-4.3; Figures 4.1-4.5.

Recapitulation

18.2

An element of fluid consisting of particles with velocity $\vec{v}$ and local density N is characterized by its three-dimensional <u>flow vector</u>

$$N\vec{v} \equiv \mathcal{J}$$

The rate at which such particles cross the area spanned by vectors $\vec{A}_1$ and $\vec{A}_2$ is the particle current

$$N\vec{v}\cdot\vec{A}_1\times\vec{A}_2 = \frac{d\#}{dt} \quad \left(=\frac{(\text{particles})}{(\text{time})}\right).$$

Relative to a chosen basis, say $\{e_i\}$ and its dual basis $\{\omega^j\}$, each of the vectors of this vector triple product is a linear combination the basis vectors

$$\vec{v} = e_i v^i; \quad \vec{A}_1 = e_j\,\omega^j(\vec{A}_1); \quad \vec{A}_2 = e_k\,\omega^k(\vec{A}_2)$$

The multilinearity of the vector triple product is brought into focus by the expansion of its determinantal form

$$N\vec{v}\cdot\vec{A}_1\times\vec{A}_2 = \begin{vmatrix} Nv^1 & Nv^2 & Nv^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix}$$

$$\begin{aligned}
&= Nv^1\left[\omega^2(\vec{A}_1)\,\omega^3(\vec{A}_2) - \omega^3(\vec{A}_1)\,\omega^2(\vec{A}_2)\right] \\
&+ Nv^2\left[\omega^3(\vec{A}_1)\,\omega^1(\vec{A}_2) - \omega^1(\vec{A}_1)\,\omega^3(\vec{A}_2)\right] \\
&+ Nv^3\left[\omega^1(\vec{A}_1)\,\omega^2(\vec{A}_2) - \omega^2(\vec{A}_1)\,\omega^1(\vec{A}_2)\right] \\
&= \left\{Nv^1\left[\omega^2\otimes\omega^3 - \omega^3\otimes\omega^2\right] + Nv^2\left[\omega^3\otimes\omega^1 - \omega^1\otimes\omega^3\right] + Nv^3\left[\omega^1\otimes\omega^2 - \omega^2\otimes\omega^1\right]\right\}(\vec{A}_1, \vec{A}_2)
\end{aligned}$$

Introduce the <u>antisymmetric wedge product</u>.

$$\omega^j\wedge\omega^k = \omega^j\otimes\omega^k - \omega^k\otimes\omega^j$$

and the totally antisymmetric Levi-Civita symbol (18.3)

$$\epsilon_{ijk} = \begin{cases} +1 & \text{whenever } (i\,j\,k) \text{ is an even permutation of } (123) \\ -1 & \text{whenever } (i\,j\,k) \text{ is an odd permutation of } (123) \\ 0 & \text{for any repeated indices.} \end{cases}$$

This line of reasoning applied to this pair of concepts exposes the tensorial multilinear map at the root of the vector triple product. Indeed, by applying the inner product to the flow vector $\not{f} = N\vec{v}$ and the area normal $\vec{A}_1\times\vec{A}_2$,

$$N\vec{v}\cdot\vec{A}_1\times\vec{A}_2 = \{Nv^1\,\epsilon_{1jk}\,\omega^j\wedge\omega^k/2! + Nv^2\,\epsilon_{2jk}\,\omega^j\wedge\omega^k/2! + Nv^3\,\epsilon_{3jk}\,\omega^j\wedge\omega^k/2!\}(\vec{A}_1,\vec{A}_2)$$
$$= Nv^i\,\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!\,(\vec{A}_1,\vec{A}_2)$$

one uncovers the flux tensor in three dimensional space,

$$*\not{f} = Nv^i\,\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!$$

This is an antisymmetric tensor of rank $\binom{0}{2}$ whose components are $\{Nv^i\epsilon_{ijk} : j,k=1,2,3\}$. In the terminology of modern mathematics it is a "scalar-valued 2-form".

Comment 1:

The development in the line of reasoning from the flow vector $\not{f} = N\vec{v}$ to the flux tensor $*\not{f} = Nv^i\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!$ is a non-trivial step forward. This is because $Nv^i\epsilon_{ijk}\,\omega^j\wedge\omega^k(\vec{A}_1,\vec{A}_2)$ lends itself to being generalized to 4-d spacetime, while $N\vec{v}\cdot\vec{A}_1\times\vec{A}_2$ does not. Indeed, the extension to 4-d spacetime consists of identifying the current 4-vector $\mathbb{J} = N\mathbf{u} = Nu^\mu \mathbf{e}_\mu$, and then by the above 3-d line of reasonin identify the density-flux tensor $*\mathbb{J} = Nu^\mu\,\epsilon_{\mu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma/3!$. This is a tensor

18.4

of rank $\binom{0}{3}$. It is a scalar-valued 3-form, which when evaluated on some triad of spacetime vectors, say A_1, A_2, A_3, yields

$$*\mathbb{J}(A_1, A_2, A_3) = N u^\mu \epsilon_{\mu\alpha\beta\gamma}\, \omega^\alpha \wedge \omega^\beta \wedge \omega^\gamma / 3!\,(A_1, A_2, A_3).$$

This is a scalar, the number of particles in the spacetime 3-d volume spanned by the spacetime vectors A_1, A_2, and A_3.

Comment 2

The rate at which particles cross the area $\vec{A}_1 \times \vec{A}_2$ spanned by the pair of 3-d vectors $\vec{A}_1$ and $\vec{A}_2$,

$$\frac{d\#}{dt} = N\vec{v}\cdot \vec{A}_1 \times \vec{A}_2\,,$$

comes from the inner product of its two factors, the particle flow vector $N\vec{v} = Nv^m e_m$ and the normal to the area spanned by $\vec{A}_1$ and $\vec{A}_2$. On the other hand, that particle rate also equals

$$\frac{d\#}{dt} = N v^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2!\,(\vec{A}_1, \vec{A}_2).$$

There is no explicit reference to any inner product operation. In spite of this, with a given inner product and a slight reformulation in terms of the metric and its inverse,

$$e_m \cdot e_\ell\, g^{\ell i} = \delta_m^{\ i}\,,$$

the implicit inner product form of the particle flux tensor is rendered manifestly explicit:

$$\frac{d\#}{dt} = \underbrace{N v^m e_m}_{N\vec{v}} \cdot e_\ell\, g^{\ell i} \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2!\,(\vec{A}_1, \vec{A}_2)$$

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The virtue of this inner product form becomes evident upon comparing it with it original vector triple product form

$$\frac{d\#}{dt} = N\vec{v}\cdot \vec{A}_1\times\vec{A}_2 .$$

Observe that the two are equal for all flow vectors $N\vec{v}$. Consequently, conclude that

$$\vec{A}_1\times\vec{A}_2 = e_\ell\, g^{\ell i}\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!\,\langle\vec{A}_1,\vec{A}_2\rangle$$
$$\equiv e_\ell\, d^2\Sigma^\ell\,(\vec{A}_1,\vec{A}_2)$$

The tensor $\boxed{e_\ell\, d^2\Sigma^\ell = e_\ell\, g^{\ell i}\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!}$

is of rank $\binom{1}{2}$.

- It belongs to the tensor space $V\otimes V^*\wedge V^* \subset V\otimes V^*\otimes V^*$
- It is a vector-valued two-form
- It is a bilinear map which maps pairs of vectors into vectors:

$$\text{"}\times\text{"} = e_\ell\, d^2\Sigma^\ell : \; V\times V \longrightarrow V$$

$$(A_1, A_2) \rightsquigarrow e_\ell\, d^2\Sigma^\ell\,(A_1,A_2) = e_\ell\, g^{\ell i}\epsilon_{ijk}\,\omega^j\wedge\omega^k/2!\,(\vec{A}_1,\vec{A}_2) = \vec{A}_1\times\vec{A}_2$$

- It can be generalized to 4-d spacetime, while the 3-d cross product cannot.

Comment 3:

The extension to 4-d spacetime consists of identifying the current 4-vector $\mathbb{J} = N\mathbb{u} = Nu^\mu \mathbb{e}_\mu$, and then by the above 3-d line of reasoning identify the density-flux tensor

$$*\mathbb{J} = N\mathbb{u}\cdot\mathbb{e}_\mu\, g^{\mu\nu}\,\epsilon_{\nu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma/3!\,.$$
$$\equiv N\mathbb{u}\cdot\mathbb{e}_\mu\, d^3\Sigma^\mu$$

Here
$$\mathbb{e}_\mu\, d^3\Sigma^\mu = \mathbb{e}_\mu\, g^{\mu\nu}\,\epsilon_{\nu\alpha\beta\gamma}\,\omega^\alpha\wedge\omega^\beta\wedge\omega^\gamma/3!$$
is a vector-valued 3-form, a tensor of rank $\binom{1}{3}$

I. Flux tubes as tensors: magnetic, electric, particle (18.6)

Consider a scalar two-form, i.e. an antisymmetric rank $\binom{0}{2}$ tensor expressed relative to a 3-d the basis $\{\omega^1, \omega^2, \omega^3\}$ dual to an appropriately chose basis $\{e_1, e_2, e_3\}$ for $V = E^3$,

$$\underset{\sim}{j} = j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! = j^i \epsilon_{i|jk|}\, \omega^j \wedge \omega^k .$$

Evaluate it on a pair of area spanning vectors $\vec{A}_1$ and $\vec{A}_2$ and obtain the number

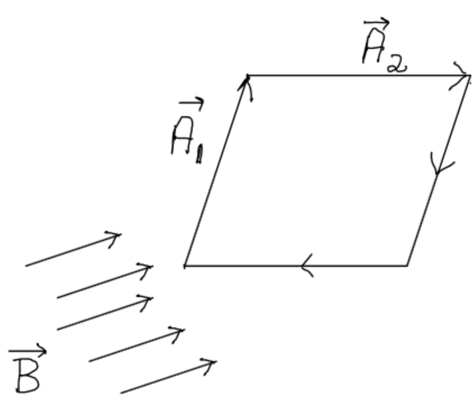
$$\underset{\sim}{j}(\vec{A}_1, \vec{A}_2) = j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2!\, (\vec{A}_1, \vec{A}_2) \quad \left(= \vec{J} \cdot \vec{A}_1 \times \vec{A}_2 ;\ \vec{J} = j^i e_i\right)$$

$$\equiv |g|^{1/2} \det \begin{vmatrix} j^1 & j^2 & j^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix}$$

where $g \equiv \det[g_{mn}]$, the determinant of the components of the metric of E^3.

The geometrical and physical meaning of $\underset{\sim}{j}$ is that it assigns a "flux tube" to E^3.
The driving force for introducing this kind of mathematical object comes, among others, from e.m. and fluid dynamics.

a) Let $\vec{B} : \{B^1, B^2, B^3\}$ be the magnetic field in a small neighborhood of a given point and $(\vec{A}_1, \vec{A}_2)$ a pair of vectors that span a small current loop in that neighborhood.



(18.7)

Figure 18.1: Magnetic field intercepted by a loop spanned by vectors $\vec{A}_1$ and $\vec{A}_2$

Because of experimental observations due to Faraday, the magnetic field is also known as "magnetic flux density", which is measured in units of Weber/area [m.k.s] or Maxwell/area [c.g.s]. For this and for modern (i.e. post W.W.II) mathematical reasons, "magnetic flux density" is mathematized by the rank $\binom{0}{2}$ tensor

$$B^i \epsilon_{ijk}\, \omega^j \wedge \omega^k/2! \equiv \mathbf{B}$$

Evaluate it on the oriented pair of loop vectors $(\vec{A}_1, \vec{A}_2)$. The resulting number,

$$\mathbf{B}(\vec{A}_1, \vec{A}_2) = |g|^{1/2} \det \begin{vmatrix} B^1 & B^2 & B^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix}, \qquad (18.1)$$

the amount of magnetic flux (measured in units of Webers or Maxwells).
Consider the closed boundary curve of the area spanned by $(\vec{A}_1, \vec{A}_2)$. From every point on this curve draw the lines tangent to the B-field. These tangent lines will generate a surface which is a tube which contains the well-defined amount $\mathbf{B}(\vec{A}_1, \vec{A}_2)$ of magnetic flux.

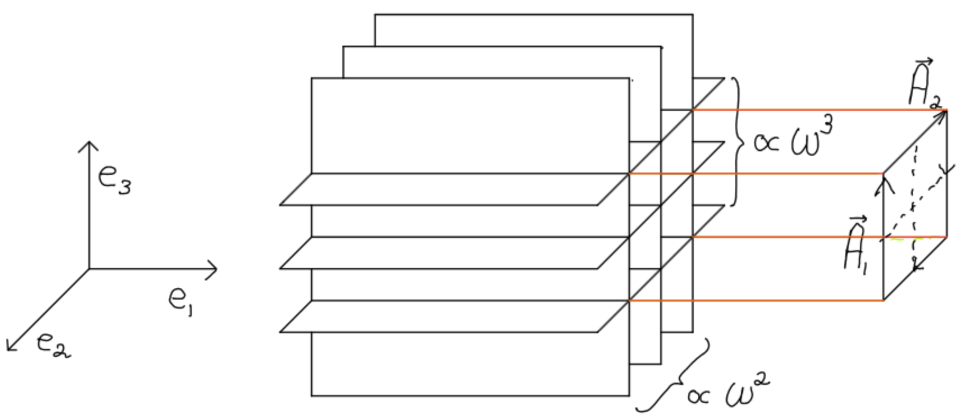


Figure 18.2: The magnetic flux density $B = B^1\,\omega^2\wedge\omega^3$ (18.8)
is a flux tube structure whose intercept with the area spanned by $(\vec{A}_1,\vec{A}_2)$ determines a fluxtube which contains $B(\vec{A}_1,\vec{A}_2)$ amount of magnetic flux.

b) Let $\vec{E}:\{E^1,E^2,E^3\}$ be the electric field in a small neighborhood of a given point and $(\vec{A}_1,\vec{A}_2)$ a pair of vectors that span a small capacitor plate in that neighborhood.

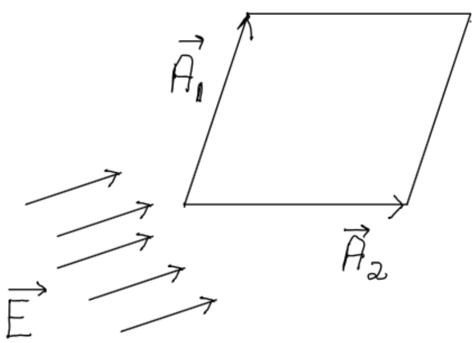


Figure 18.3: Electric field lines terminate at a conducting plate spanned by vectors $\vec{A}_1$ and $\vec{A}_2$

The electric field (a.k.a. "electric intensity" or "electric field strength") is a force field. It is measured in terms of "force per charge" (newtons/coulomb [m.g.s.] or dynes/statcoulomb [c.g.s.]) and conceptualized in the form of Faraday's lines of force per area. In the light of modern post W.W.II mathematics, the electric field is mathematized in the form of Faraday's "lines of force per area", a scalar-valued two-form

$$E = E^i\,\epsilon_{ijk}\,\omega^j\wedge\omega^k/2! \qquad (18.2)$$

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The number of lines of force intercepted by the area spanned by the pair of oriented vectors $(\vec{A}_1, \vec{A}_2)$ is

$$E(\vec{A}_1, \vec{A}_2) = |g|^{1/2} \det \begin{vmatrix} E^1 & E^2 & E^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix}, \qquad (18.3)$$

Compare Eq. (18.3) with Eq. (18.1) on page 18.3. From it and from the remarks following it, conclude that Eq. (18.2) has the geometrical structure of a sum of flux tubes of the type depicted in Figure 18.2, and that Eq. (18.3) is the number lines of force contained in them.

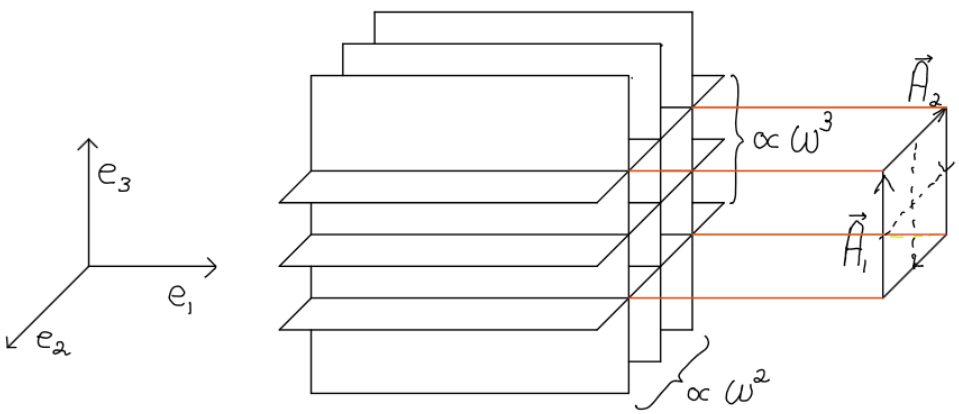

Figure 18.4: Flux tube structure of a simple electric electric force line density $E = E^1 \epsilon_{123}\, \omega^2 \wedge \omega^3$.
The $(\vec{A}_1, \vec{A}_2)$ determined flux tube contains $E(\vec{A}_1, \vec{A}_2)$ lines of electric force.

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c) Let $\vec{j} = N\vec{v} : \{Nv^1, Nv^2, Nv^3\}$ be the flux vector of an element of fluid in a small neighborhood of a point in the fluid. Each of its fluid particles traces out its particle trajectory.
As conceptualized by Maxwell in 1861, these trajectories give rise to the concept of a flux tube by the following line of reasoning:

"If upon any surface which cuts the lines of fluid motion we draw a closed curve, and if from every point of this curve we draw lines of motion, these lines of motion will generate a tubular surface which we may call a [flux] tube of fluid motion."

Consider the closed curve to be in the shape of a parallelogram spanned by $(\vec{A}_1, \vec{A}_2)$. The number of lines of motion inside that tubular flux tube is

$$\vec{j} \cdot \vec{A}_1 \times \vec{A}_2 \equiv |g|^{1/2} \det \begin{vmatrix} j^1 & j^2 & j^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \end{vmatrix}$$

$$= j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! \,(\vec{A}_1, \vec{A}_2)$$

The number of lines of motion per tubular cross section is

$$j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! = N v^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! \equiv *j$$

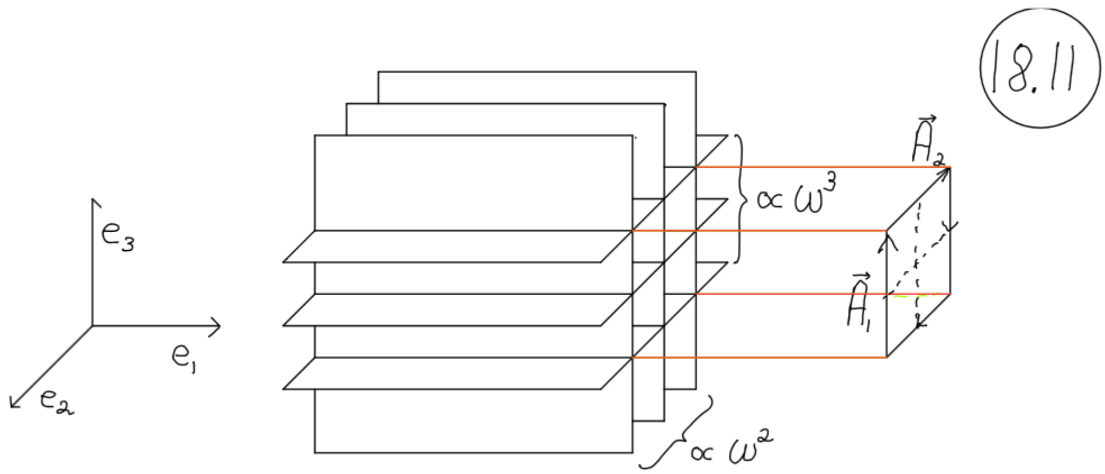

Figure 18.5: Flux tube structure of $*\mathbf{J} = N v' \epsilon_{123}\, \omega^2 \wedge \omega^3/2!$, a simple density of lines of motion.
The $(\vec{A}_1, \vec{A}_2)$-determined flux tube contains $*\mathbf{J}(\vec{A}_1, \vec{A}_2)$ particle lines of motion.