LECTURE 18

18.1

I. Flux tube structures

a) Magnetic flux density

b) Flux density of electric force lines

c) Flux density of lines of particle motion

In MTW Sections 4.2-4.3; Figures 4.1-4.5.

I. Flux tubes as tensors: magnetic, electric, particle (18.2)

Consider a scalar two-form, i.e. an antisymmetric rank $\binom{0}{2}$ tensor expressed relative to a 3-d the basis $\{\omega^1, \omega^2, \omega^3\}$ dual to an appropriately chose basis $\{e_1, e_2, e_3\}$ for $V = E^3$,

$$\underset{\sim}{j} = j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! = j^i \epsilon_{i|jk|}\, \omega^j \wedge \omega^k.$$

Evaluate it on a pair of area spanning vectors $\vec{A}_1$ and $\vec{A}_2$ and obtain the number

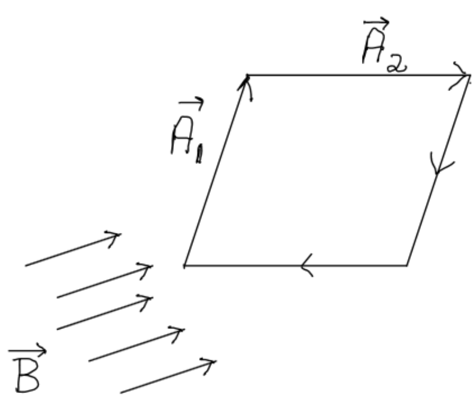
$$\underset{\sim}{j}(\vec{A}_1, \vec{A}_2) = j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2!\, (\vec{A}_1, \vec{A}_2) \quad \left(= \vec{j} \cdot \vec{A}_1 \times \vec{A}_2 ;\ \vec{j} = j^i e_i \right)$$

$$\equiv |g|^{1/2} \det \begin{vmatrix} j^1 & j^2 & j^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix}$$

where $g \equiv \det[g_{mn}]$, the determinant of the components of the metric of E^3.

The geometrical and physical meaning of $\underset{\sim}{j}$ is that it assigns a "<u>flux tube</u>" to E^3. The driving force for introducing this kind of mathematical object comes, among others, from e.m. and fluid dynamics.

a) Let $\vec{B}: \{B^1, B^2, B^3\}$ be the magnetic field in a small neighborhood of a given point and $(\vec{A}_1, \vec{A}_2)$ a pair of vectors that span a small current loop in that neighborhood.



18.3

Figure 18.1: Magnetic field intercepted by a loop spanned by vectors $\vec{A}_1$ and $\vec{A}_2$

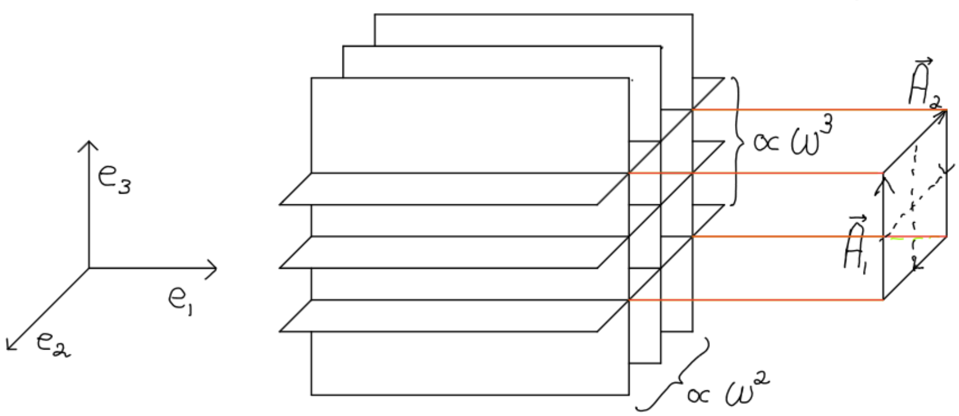
Because of experimental observations due to Faraday, the magnetic field is also known as "magnetic flux density," which is measured in units of Weber/area [m.k.s] or Maxwell/area [c.g.s]. For this and for modern (i.e. post W.W.II) mathematical reasons, "magnetic flux density" is mathematized by the rank $\binom{0}{2}$ tensor

$$B^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! \equiv \mathbb{B}$$

Evaluate it on the oriented pair of loop vectors $(\vec{A}_1, \vec{A}_2)$. The resulting number,

$$\mathbb{B}(\vec{A}_1, \vec{A}_2) = |g|^{1/2} \det \begin{vmatrix} B^1 & B^2 & B^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix}, \qquad (18.1)$$

the amount of magnetic flux (measured in units of Webers or Maxwells). Consider the closed boundary curve of the area spanned by $(\vec{A}_1, \vec{A}_2)$. From every point on this curve draw the lines tangent to the B-field. These tangent lines will generate a surface which is a tube which contains the well-defined amount $\mathbb{B}(\vec{A}_1, \vec{A}_2)$ of magnetic flux.



(18.4)

Figure 18.2: The magnetic flux density $B = B^1 \omega^2 \wedge \omega^3$ is a flux tube structure whose intercept with the area spanned by $(\vec{A}_1, \vec{A}_2)$ determines a flux tube which contains $B(\vec{A}_1, \vec{A}_2)$ amount of magnetic flux.

b) Let $\vec{E}: \{E^1, E^2, E^3\}$ be the electric field in a small neighborhood of a given point and $(\vec{A}_1, \vec{A}_2)$ a pair of vectors that span a small capacitor plate in that neighborhood.

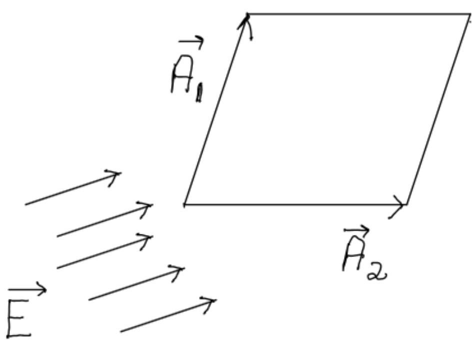


Figure 18.3: Electric field lines terminate at a conducting plate spanned by vectors $\vec{A}_1$ and $\vec{A}_2$

The electric field (a.k.a. "electric intensity" or "electric field strength") is a force field. It is measured in terms of "force per charge" (newtons/coulomb [m.g.s.] or dynes/ statcoulomb [c.g.s.]) and conceptualized in the form of Faraday's lines of force per area. In the light of modern post W.W. II mathematics, the electric field is mathematized in the form of Faraday's "lines of force per area", a scalar-valued two-form

$$E = E^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! \qquad (18.2)$$

18.5

The number of lines of force intercepted by the area spanned by the pair of oriented vectors $(\vec{A}_1, \vec{A}_2)$ is

$$E(\vec{A}_1, \vec{A}_2) = |g|^{1/2} \det \begin{vmatrix} E^1 & E^2 & E^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_2) & \omega^2(\vec{A}_2) & \omega^3(\vec{A}_2) \end{vmatrix}, \qquad (18.3)$$

Compare Eq. (18.3) with Eq. (18.1) on page 18.3. From it and from the remarks following it, conclude that Eq. (18.2) has the geometrical structure of a sum of flux tubes of the type depicted in Figure 18.2, and that Eq. (18.3) is the number lines of force contained in them.

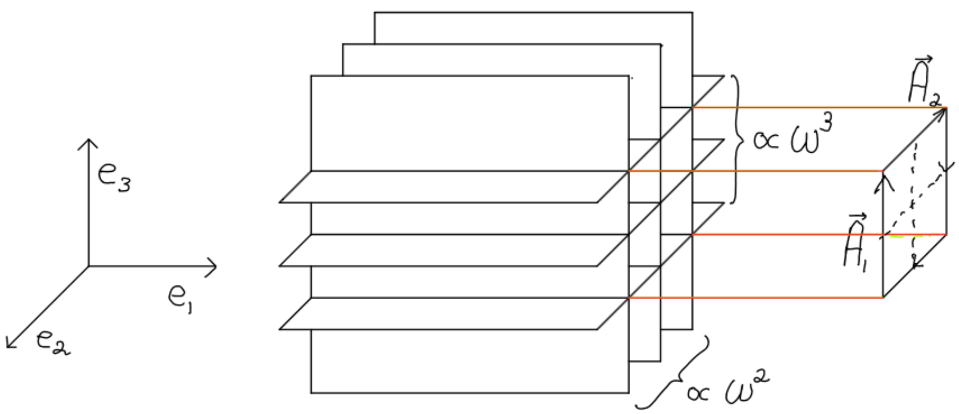


Figure 18.4: Flux tube structure of a simple electric electric force line density $E = E^1 \epsilon_{123}\, \omega^2 \wedge \omega^3$.
The $(\vec{A}_1, \vec{A}_2)$ determined flux tube contains $E(\vec{A}_1, \vec{A}_2)$ lines of electric force.

18.6

c) Let $\vec{j} = N\vec{v} : \{Nv^1, Nv^2, Nv^3\}$ be the flux vector of an element of fluid in a small neighborhood of a point in the fluid. Each of its fluid particles traces out its particle trajectory.

As conceptualized by Maxwell in 1861, these trajectories give rise to the concept of a <u>flux tube</u> by the following line of reasoning:

"If upon any surface which cuts the lines of fluid motion we draw a closed curve, and if from every point of this curve we draw lines of motion, these lines of motion will generate a tubular surface which we may call a <u>[flux] tube of fluid motion</u>."

Consider the closed curve to be in the shape of a parallelogram spanned by $(\vec{A}_1, \vec{A}_2)$. The number of lines of motion inside that tubular flux tube is

$$\vec{j} \cdot \vec{A}_1 \times \vec{A}_2 \equiv |g|^{1/2} \det \begin{vmatrix} j^1 & j^2 & j^3 \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \\ \omega^1(\vec{A}_1) & \omega^2(\vec{A}_1) & \omega^3(\vec{A}_1) \end{vmatrix}$$

$$= j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! \,(\vec{A}_1, \vec{A}_2)$$

The number of lines of motion per tubular cross section is

$$j^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! = N v^i \epsilon_{ijk}\, \omega^j \wedge \omega^k / 2! \equiv *\mathcal{J}$$

(18.7)

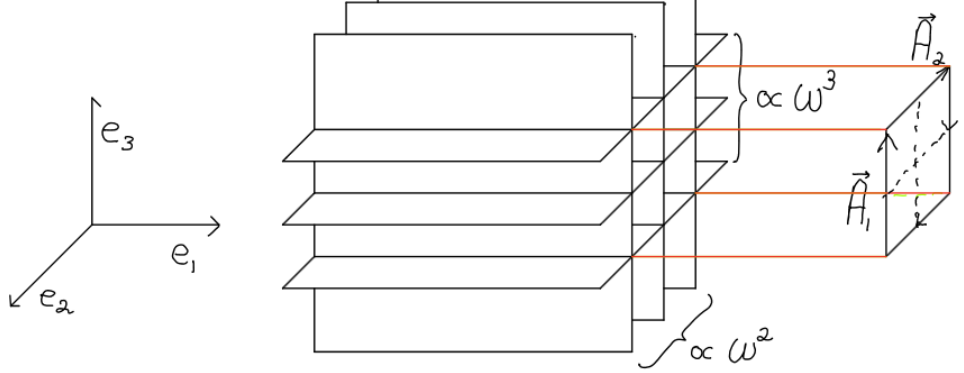


Figure 18.5: Flux tube structure of $*\mathcal{J} = N v^1 \epsilon_{123}\, \omega^2 \wedge \omega^3 / 2!$, a simple density of lines of motion.
The $(\vec{A}_1, \vec{A}_2)$-determined flux tube contains $*\mathcal{J}(\vec{A}_1, \vec{A}_2)$ particle lines of motion.