LECTURE 2

2.1

Key Idea: Invariance of the Interval

T-W $\underbrace{3.6,\ 3.7,\ 3.8}_{2^{nd}\text{ edition}}\ \underbrace{(1.5)}_{1^{st}\text{ edition}}$

LECTURE 2

The most important consequence of the Principle of Relativity is the Invariance of the Interval. (2.2a)

The "interval" between two point-events in spacetime is what in Euclidean space is the "distance" between two points. The "invariance" expresses the inertial frame independence of this interval.

Given two points, O and P, in Euclidean space.

Q: How does one measure the distance $d(O,P)$ between O and P?

A: There are two ways of measuring distance:

1. Direct measurement. (2.2b)

Apply and then count a sequence of standard meter rods to the displacement $\overrightarrow{OP}$.

Figure 2.0a: Distance $d(O,P)$ as measured by counting the number of standard meter sticks between O and P.

2. Introduce a Cartesian coordinate system, say (x,y), and use the Pythagorean theorem.
Point P is represented by

$$\big(x(P), y(P)\big)$$

relative to this chosen coordinate system. The distance between its origin O and the point P is the square root of the Pythagorean squared distance

$$x^2+y^2 \equiv r^2.$$

If one introduces a different coordinate system, say $(\bar{\bar{x}},\bar{\bar{y}})$ as in Figure 2.0b, then point P is represented differently, namely by

$$\big(\bar{\bar{x}}(P), \bar{\bar{y}}(P)\big).$$

Nevertheless, the Pythagorean distance remains the same, (2.2c)

$$x^2+y^2=\bar{\bar{x}}^2+\bar{\bar{y}}^2=r^2;$$

even though $x\neq\bar{\bar{x}}$, $y\neq\bar{\bar{y}}$.

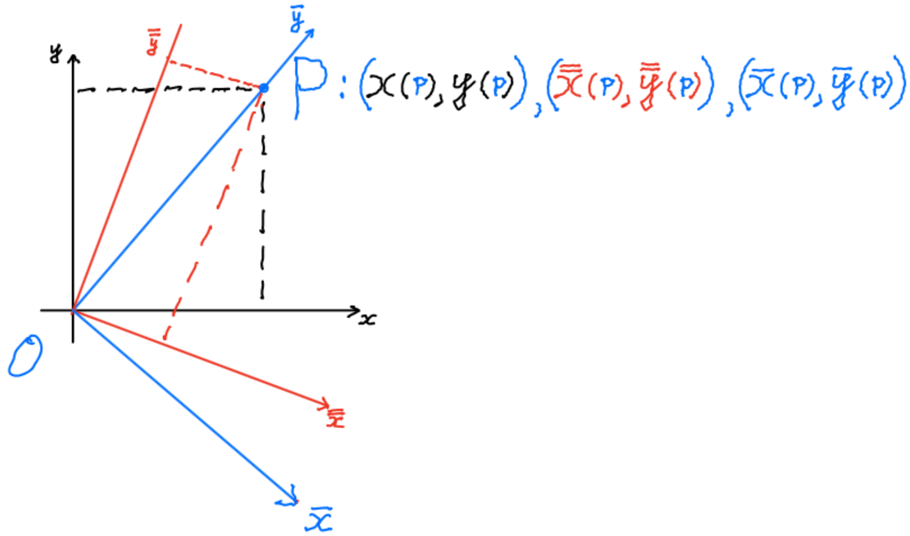

Figure 2.06: Three different surveyors with their three different coordinate systems (x,y), $(\bar{x},\bar{y})$, $(\bar{\bar{x}},\bar{\bar{y}})$ observe the point P in terms of three different pairs of measured coordinate components $(x(P), y(P))$, $(\bar{\bar{x}}(P), \bar{\bar{y}}(P))$, and $(\bar{x}(P), \bar{y}(P))$.

Each pair is called a <u>coordinate representative</u> or <u>coordinate representation</u> of the point P. In spite of their differences, the coordinatives have a common feature, namely their squared Pythagorean distance from the origin:

$$x^2(P)+y^2(P)=\bar{\bar{x}}^2(P)+\bar{\bar{y}}^2(P)=\bar{x}^2(P)+\bar{y}^2(P).$$

(2.3)

SUMMARY

In Euclidean space the distance between two points is independent of the relative orientation of the axes of different coordinate systems.

Similarly in spacetime the interval between two events recorded in different inertial frames is independent of their relative motion.

More precisely one has the following

Theorem (Invariance of the interval) (2.4)

GIVEN: (a) two events in spacetime,

$(t_1, \underset{\sim}{r}_1)$ and $(t_2, \underset{\sim}{r}_2)$

(b) Let two observers S and $\bar{S}$ measure the separation between these events in their respective inertial frames:

S measures $(\Delta t, \Delta \underset{\sim}{r})$ and

$\bar{S}$ measures $(\overline{\Delta t}, \overline{\Delta \underset{\sim}{r}})$

CONCLUSION:

$$\boxed{\text{The P. of R.} + \text{isotropy of space}} \Rightarrow \boxed{(\Delta t)^2 - (\Delta \underset{\sim}{r})^2 = (\overline{\Delta t})^2 - (\overline{\Delta \underset{\sim}{r}})^2}$$

The proof is a 3 step process of deductive reasoning.

2.5

Comments

(i) The quantity $(\Delta t)^2 - (\Delta \underset{\sim}{r})^2 \equiv (\Delta \tau)^2$ is called the (squared) <u>interval</u> between events 1 and 2.

(ii) The interval is an <u>invariant</u> because it is the same relative to all inertial frames.

(iii) We have defined time as distance traveled by light

$$\left.\begin{aligned} t &= c\, t_{\text{conv.}} \\ \bar{t} &= c\, \bar{t}_{\text{conv.}} \\ \bar{\bar{t}} &= c\, \bar{\bar{t}}_{\text{conv.}} \end{aligned}\right\} \quad c = 3\times 10^{10}\ \text{cm/sec} = 3\times 10^{8}\ \frac{\text{m}}{\text{sec}}$$

because c, the measured speed of light, is the same in all inertial frames.

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For example:

$\Delta t = 1$ meter corresponds to

$\Delta t_{conv} = 3 \times 10^{-9}$ sec $= 3$ nanoseconds

because $c\Delta t_{conv} = 3\times 10^{8} \frac{m}{sec} \times 3\times 10^{-9} sec = 1$ meter.

(End of Comments)

The deductive line of reasoning consists of first following the trajectory of a pulse of light as measured in the LAB frame and in a ROCKET frame moving relative to the LAB, then of expressing the longitudinal measurements in terms of those tranverse to the relative motion, and finally of arriving the invariance from the P. of R.

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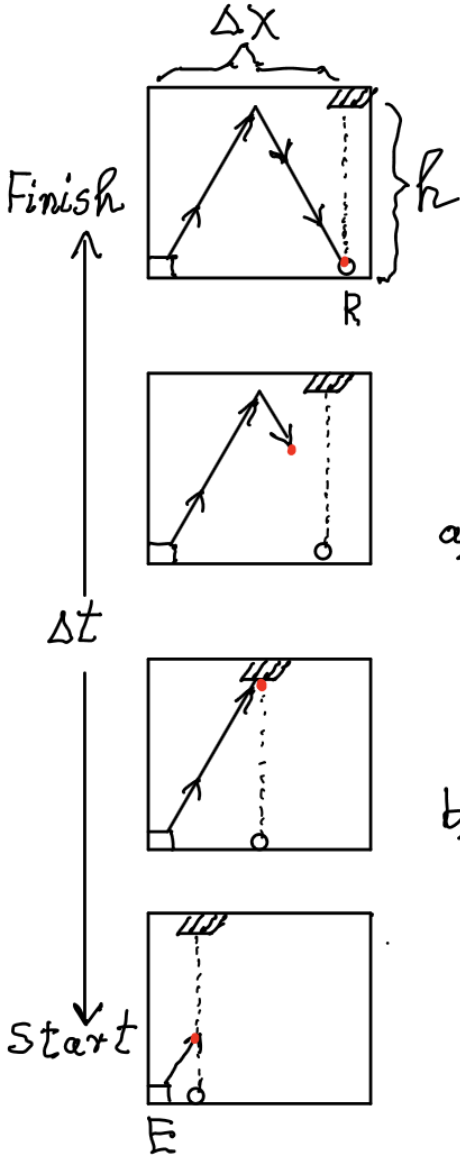


Step I:

Elapsed time between emission event E and reception event R

a) as determined in the LAB frame

$$\Delta t = \Delta t_{conv} c = 2\sqrt{\left(\frac{\Delta x}{2}\right)^2 + h^2}$$

$$\Delta X = \Delta X$$

b) as determined in the ROCKET frame:

$$\overline{\Delta t} = \overline{\Delta t}_{conv} c = 2\bar{h}$$

$$\overline{\Delta X} = 0$$

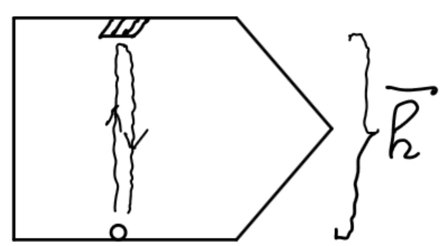


Figure 2.0c:

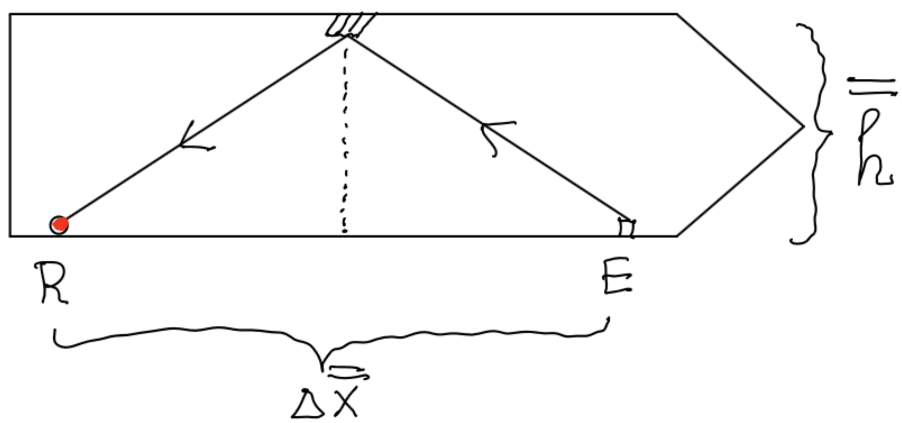
Rocket mirror (= ш) and clock (= o) moving in the lab frame (= ⌑)

c) as determined in the (2.8)

SUPER ROCKET frame:

$$\bar{\bar{\Delta t}} = \bar{\bar{\Delta t}}_{conv}\, c = 2\sqrt{\left(\frac{\bar{\bar{\Delta x}}}{2}\right)^2 + \bar{\bar{h}}^2}$$

$$\bar{\bar{\Delta x}} = \bar{\bar{\Delta X}}$$



Comment:

We have used

(i) the <u>isotropy</u> of space in a) and c)

(ii) the Principle of Relativity:

the law of light propagation, in particular the speed of light, is the <u>same</u> in <u>all</u> inertial frames.

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Step II:

Upon squaring and subtracting one finds:

$(\Delta t)^2 - (\Delta x)^2 = 4\,h^2$ in the LAB frame

$(\overline{\Delta t})^2 - (\overline{\Delta x})^2 = 4\,\overline{h}^2$ in the ROCKET frame

$(\overline{\overline{\Delta t}})^2 - (\overline{\overline{\Delta x}})^2 = 4\,\overline{\overline{h}}^2$ in the SUPER ROCKET frame

Step III:

According to the Principle of Relativity

$$h = \overline{h}$$

i.e. the length is the direction transverse to the direction of motion is the same in both frames: it is an <u>invariant</u>.

Q: Why so?

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A: Start with two wooden boards, each with a nail at height h in it

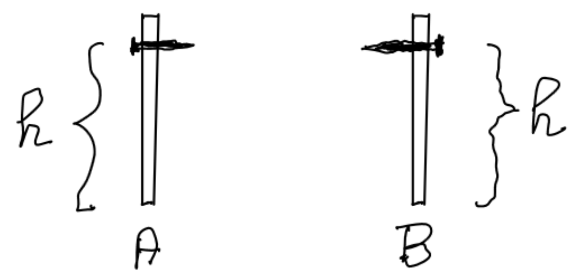


Figure 2.1:

Place board A into the ROCKET($=\bar{S}$) frame and board B into the LAB ($=S$) frame

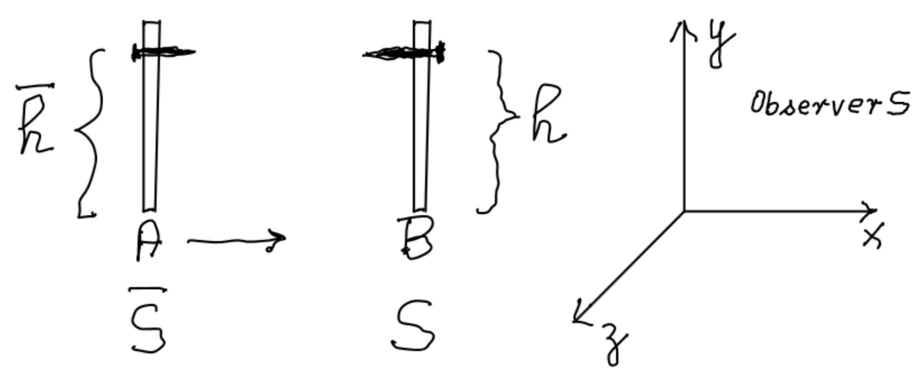


Figure 2.2:

The LAB ($=S$) observer observes $\bar{S}$ approach from the left.

A and B collide. We shall now show that the Principle of Relativity and the Isotropy of Space imply that $\bar{h} = h$.

The proof is by contradiction: <u>Assume</u> the contrary, say that $\bar{h} > h$, i.e. that, upon collision, A and B leave nail marks on each other as depicted in Figure 2.3.

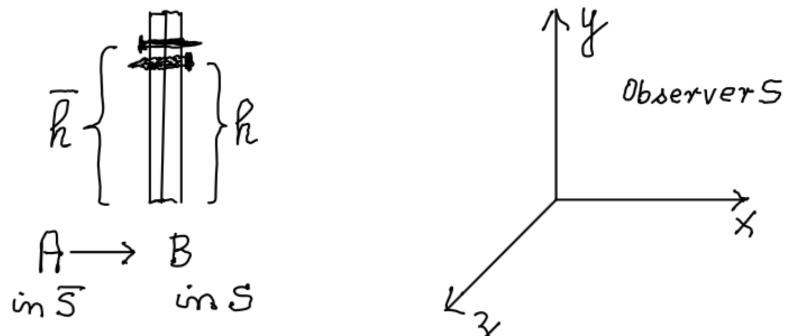


Figure 2.3: Result of collision as observed in frame S where B is static and A comes in from B's left

Next, look at the process leading up to the collision in the frame $\bar{S}$ where A is static and B comes in from A's left. For such an observer the process $\frac{A}{S} \to \frac{B}{S}$ depicted in Figure 2.2 will be observed as in Figure 2.4.

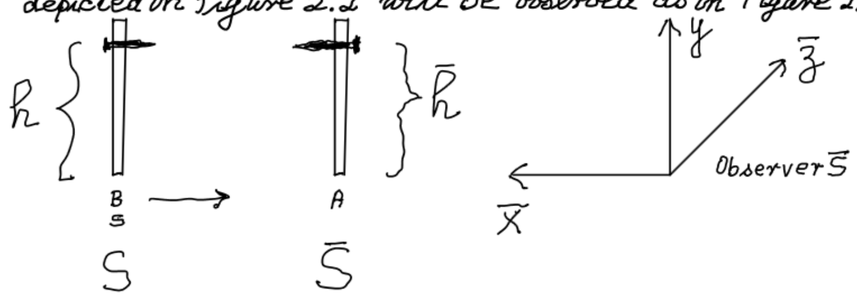


Figure 2.4: Process leading up to the collision in the frame $\bar{S}$ where A is static and B comes in from A's left.

Upon colliding, A and B will (under the assumption $\bar{h} > h$) be left with marks as depicted in Figure 5. (2.12)

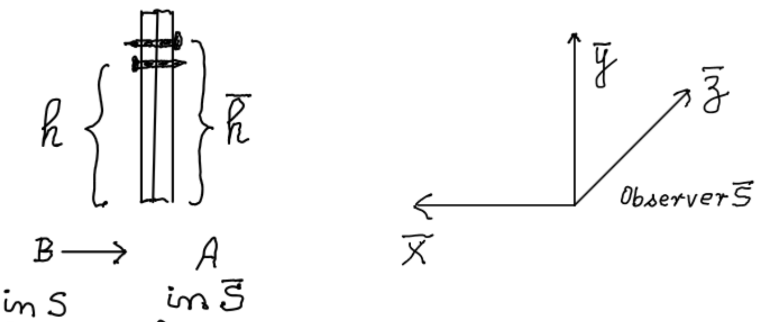


Figure 2.5: Result of collision as observed in frame $\bar{S}$ where A is static and B comes in from A's left.

Compare Figure 2.3 with Figure 2.5.

The processes giving rise to these pictures are identical: same kinds of entities, same kinds of observed motions (left to right) observed in each inertial frame. The two Figures are supposed to be identical, but they are not!

Whereas in Figure 2.3 the static object B on the right receives a mark from A <u>above</u> its nail, in Figure 2.5 static object A (also on the right) receives a mark <u>below</u> its nail. In other words, frame $\bar{S}$ is a preferred frame: it is singled out by having A leave a mark above the nail in S, B's frame. This violates the P. of R. as stated on page (1.11) of Lecture 1. Instead of leading to two different results, the collision should have led to the same result, Figure 2.6.

Thus

$$\bar{h} = h,$$

i.e. $\bar{h} > h$ would violate the P. of R. and the Isotropy of Space.

2.13

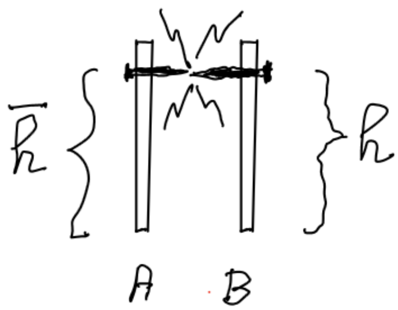


Figure 2.6: In compliance with the Principle of Relativity, the collision between A and B results in sparks.

SUMMARY:

The dimensions of moving objects transverse to the direction of relative motions are measured to be the same. This universal observation is the principle of "the invariance of transverse distance."

CONCLUSION

$$\boxed{(\Delta t)^2 - (\Delta x)^2 = (\Delta \bar{t})^2 - (\Delta \bar{x})^2 \equiv (\text{interval})^2}$$

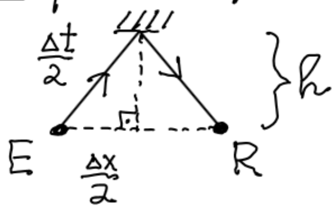
More generally:

$$\boxed{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\Delta \bar{t})^2 - (\Delta \bar{x})^2 - (\Delta \bar{y})^2 - (\Delta \bar{z})^2 \equiv (\text{interval})^2}$$

i.e. the interval between a pair of events is invariant; it is the same relative to <u>all</u> inertial reference frames.

COMMENT (2.14)

For the <u>same</u> pair of events E and R



the figures on pages (2.7) and (2.8) capture 4 central ideas:

1. $$(\text{hypotenuse})^2 - (\text{base})^2 = (\text{height})^2$$
$$\left(\tfrac{1}{2}\,\text{time sep'n}\right)^2 - \left(\tfrac{1}{2}\,\text{space sep'n}\right)^2 = \left(\tfrac{1}{2}\,\text{interval}\right)^2$$

2. Invariance of the interval

3. Invariance of the speed of light

4. $\Delta t \neq \overline{\Delta t} \neq \overline{\overline{\Delta t}}$

$\Delta x \neq \overline{\Delta x} \neq \overline{\overline{\Delta x}}$