LECTURE 20

20.1

I. Coordinate chart

II. Transition map (a.k.a. coordinate transformation)

III. Compatible charts

IV. Manifold

In MTW start reading Ch. 9, in particular §9.7

In Singer and Thorpe read the first 4 pages of Ch. 5 ("Manifolds") in Lecture Notes On Geometry and Topology.

20.2

I. Overview

Tensor calculus is a conceptual integration of (a "marriage between") multilinear algebra and multivariable calculus. The fundamental conceptual product of this integration is the differentiable manifold.

A manifold is a topological space which locally (to a "near-sighted observer") looks like a rectilinear space with a Cartesian coordinate system.

At each point of a differentiable manifold there is a tangent space, which is the vector space of all tangent vectors. They are the partial derivatives that operate on differentiable real valued functions whose domain is the manifold. The set of partial derivatives along the coordinate lines of a coordinate system form a basis of the tangent space. The set of differentials of the coordinate functions form a basis for the corresponding (dual) cotangent space.

II. Definition of a Manifold

Let M be a Hausdorff space.

[For the cognoscenti:

A topological space M consisting of its family of open sets $\mathcal{U} = \{U_1, U_2, \cdots\}$ is said to be a Hausdorff space it has enough open sets such that for any two different points s_1, s_2 in M, i.e., whenever $s_1 \neq s_2$, $\exists$ open U_1 and U_2 with $s_1 \in U_1$ and $s_2 \in U_2$ such that $U_1 \cap U_2 = \phi$, i.e. U_1 and U_2 are disjoint. Thus, in a Hausdorff space all pairs of points can be separated by non-overlapping open sets.

More details on this in Section 2.1 of "Lecture Notes On Elementary Topology" by Singer and Thorpe.]

A manifold consists of a network of interrelated concepts. The most unit-economical definition of a manifold is given by the following

Definition ("Manifold")

If for any point $P \in M$ there exists a nbhd U of P such that U is homeomorphic to an open set in R^n, then M is called an <u>n-dimensional manifold</u>.

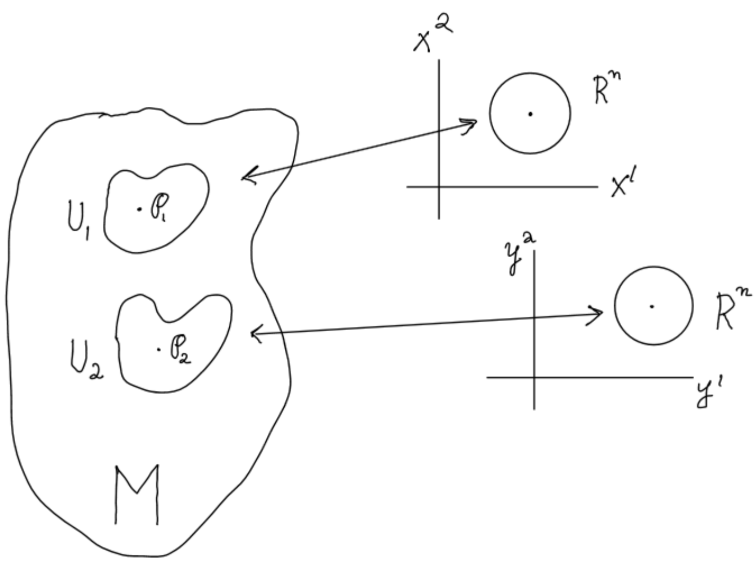

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Figure 20.1: Manifold M and two of its neighborhoods which are homeomorphic to two open set in two copies of R^n.

However, the concept of a manifold necessitates the as-yet-unmentioned constitutive property when two neighborhoods overlap as in Figure 20.2.

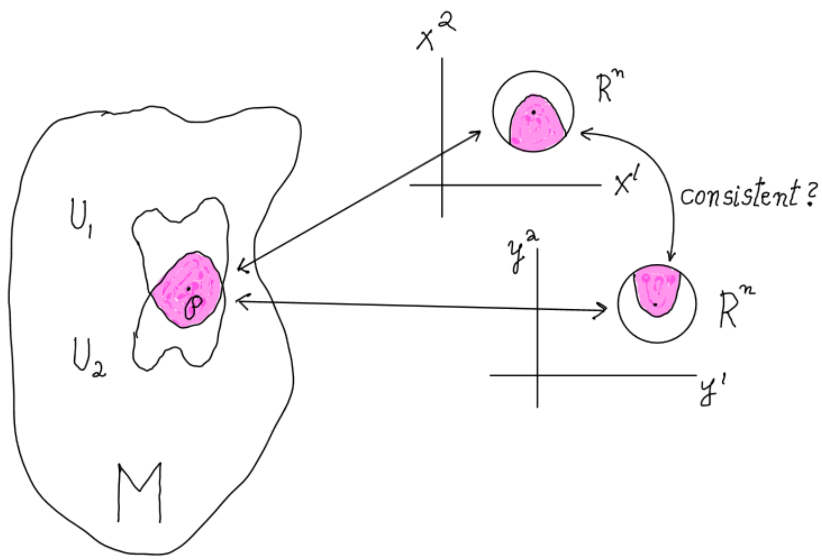

Figure 20.2: Manifold M with two overlapping neighborhoods. They necessitate consistency between the two homeomorphisms.

The consistency requirement is mathematized by making explicit what was implicit in the unit-economical definition of M.

A. Definition ("Coordinate charts")

Consider a set M. (20.5)

a) A local (= "coordinate") chart is

(i) a homeomorphic mapping φ from $U \subset M$ into a Cartesian space R^n;

$$\varphi : U \longrightarrow \varphi(U) \subset R^n$$

together with

(ii) the neighborhood U.

It is written as (φ, U)

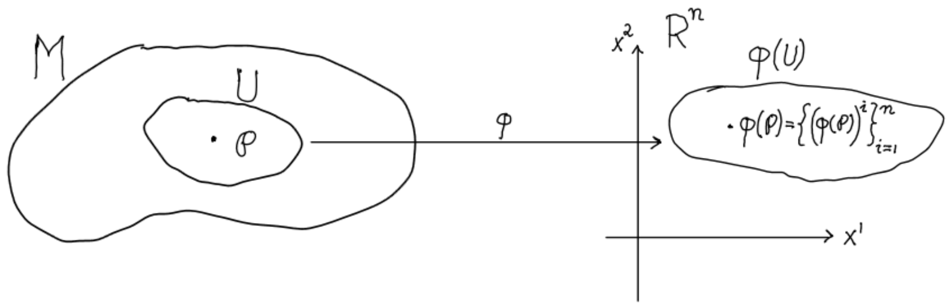

Figure 20.3: The coordinate chart (φ, U) is a homeomorphic mapping from $U \subset M$ into R^n. A point $P \in U$ gets coordinatized by

$$\varphi(P) \in R^n, \text{ i.e. by } x^i(P) = (\varphi(P))^i, \; i = 1, 2, \dots, n$$

which are the local coordinates of P relative to the local chart (a.k.a. "coordinate system") (φ, U)

b) Consider two overlapping coordinate charts (φ_1, U_1) and (φ_2, U_2), i.e. a pair for which $U_1 \cap U_2 \neq \emptyset$. This intersection has two images, one under φ_1, namely $\varphi_1(U_1 \cap U_2)$, the other under φ_2, namely $\varphi_2(U_1 \cap U_2)$. These images lie in different copies of R^n.

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The two charts, (φ_1, U_1) and (φ_2, U_2), are said to be consistent if there exists a homeomorphism between their two images of $U_1 \cap U_2$.

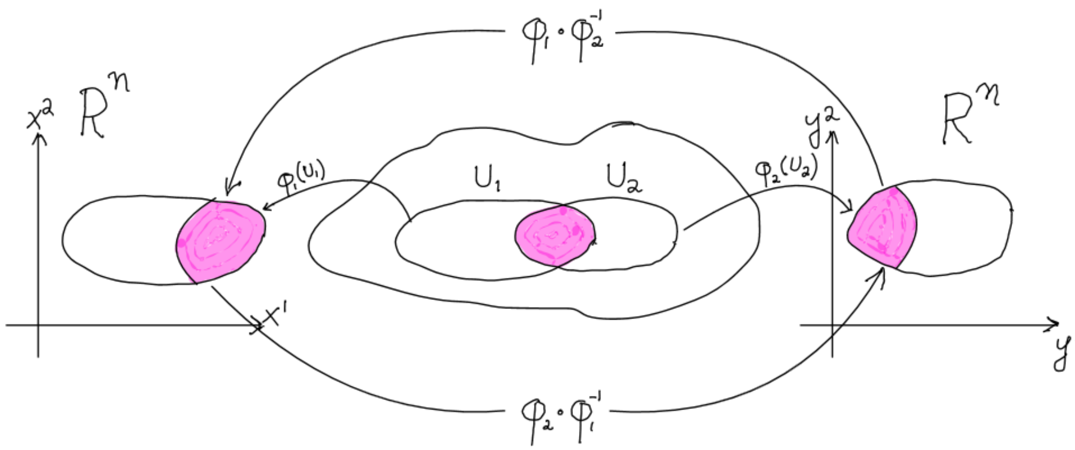

Figure 20.4: Two overlapping charts and their transition maps $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$.

These two overlapping coordinate charts define the homeomorphism

$$\varphi_1 \circ \varphi_2^{-1}\Big|_{\varphi_2(U_1 \cap U_2)} : \varphi_2(U_1 \cap U_2) \longrightarrow \varphi_1(U_1 \cap U_2)$$
$$\{y^j\} \rightsquigarrow \{x^i\}$$

and its inverse

$$\varphi \circ \varphi_1^{-1}\Big|_{\varphi_1(U_1 \cap U_2)} : \varphi_1(U_1 \cap U_2) \longrightarrow \varphi_2(U_1 \cap U_2)$$
$$\{x^i\} \rightsquigarrow \{y^j\}$$

They are coordinate transformations whose explicit forms are

$$x^i = g^i(y^1, \dots, y^n) = \left(\varphi_1 \circ \varphi_2^{-1}(y^1, \dots, y^n)\right)^i, \quad i = 1, \dots, n$$

where $(y^1, \dots, y^n) \in \varphi_2(U_1 \cap U_2)$,

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and

$$y^j = f^j(x^1,\dots,x^n) = \left(\varphi_2 \circ \varphi_1^{-1}(x^1,\dots,x^n)\right)^j, \quad j=1,\dots,n$$

where $(x^1,\dots,x^n) \in \varphi_1(U_1 \cap U_2)$.

Comment

1. $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$ are homeomorphic <u>inverses</u> of each other; they are called <u>transition maps</u>.
2. f^j and g^i are continuous functions.
3. $f^j(g^1(y^1,\dots,y^n),\dots,g^n(y^1,\dots,y^n)) = y^j$

   $g^i(f^1(x^1,\dots,x^n),\dots,f^n(x^1,\dots,x^n)) = x^i$
4. One says that the charts (φ_1, U_1) and (φ_2, U_2) are <u>C^r-compatible</u> or C^r-related if

   (i) $U_1 \cap U_2 \neq \emptyset$ and

   (ii) $f^j(x^1,\dots,x^n)$ and $g^i(y^1,\dots,y^n)$

   are C^r, i.e. have continuous partial derivatives up to order r.

   C^0 means continuous; C^∞ means all partial derivatives of all orders exist.

B.) Definition ("Atlas")

A C^r <u>n-atlas</u> $\mathcal{A}$ is a collection of C^r-related charts $\{(\varphi_1, U_1), (\varphi_2, U_2), \dots\} = \mathcal{A}$ such that

(i) $\{(\varphi_1 U_1), (\varphi_2, U_2), \dots\}$ is an open covering of M, i.e. M is the union of $U_1, U_2, \dots$: $M = \bigcup_i U_i$;

(ii) any two coordinate charts in $\mathcal{A}$ are C^r-compatible;

(iii) $\mathcal{A}$ is <u>maximal</u>, i.e. if a coordinate chart $(\tilde{U}, \tilde{\varphi})$ is C^r-compatible with all all coordinate charts in $\mathcal{A}$, then $(\tilde{U}, \tilde{\varphi}) \in \mathcal{A}$.

C.) Definition ("Manifold") 20.8

An n-dimensional C^r-manifold is a set M with the structure of a C^r n-atlas which is maximal.

Note: A maximal atlas contains all possible charts.

D.) A given atlas $\mathcal{A}$ on M is called a differentiable structure on M.

E.) If $\mathcal{A}$ is a C^∞ n-atlas is given on M, then M is called a smooth manifold.

Comment

There may exist distinct differentiable structures on a single topological (i.e. C^0) manifold. John Milnor in 1956 gave a famous example which shows that there exist nonisomorphic smooth structures on homeomorphic topological manifolds.
In particular a topological seven-sphere, S^7, admits 28 distinct differentiable structures.