LECTURE 21

21.1

I. Transition function exemplified

II. Scalar function representatives

III. Transition maps as links between overlapping charts.

I. Transition map exemplified.

Consider the 1-d manifold

$$M = S^1 = \{(x,y): x^2+y^2=1\} \leftarrow \text{"implicitly defined"}$$
$$= \{-\pi<\theta\leq\pi :\ x=\cos\theta;\ y=\sin\theta\} \leftarrow \boxed{\text{"explicitly defined"}}$$

and two of its coordinate charts (φ_N, U_N) and (φ_E, U_E).

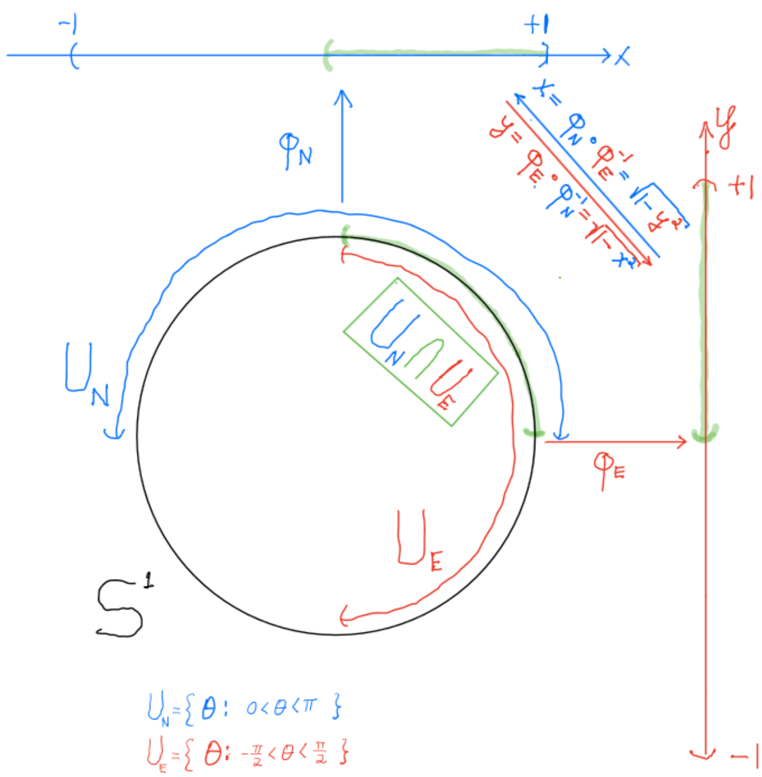
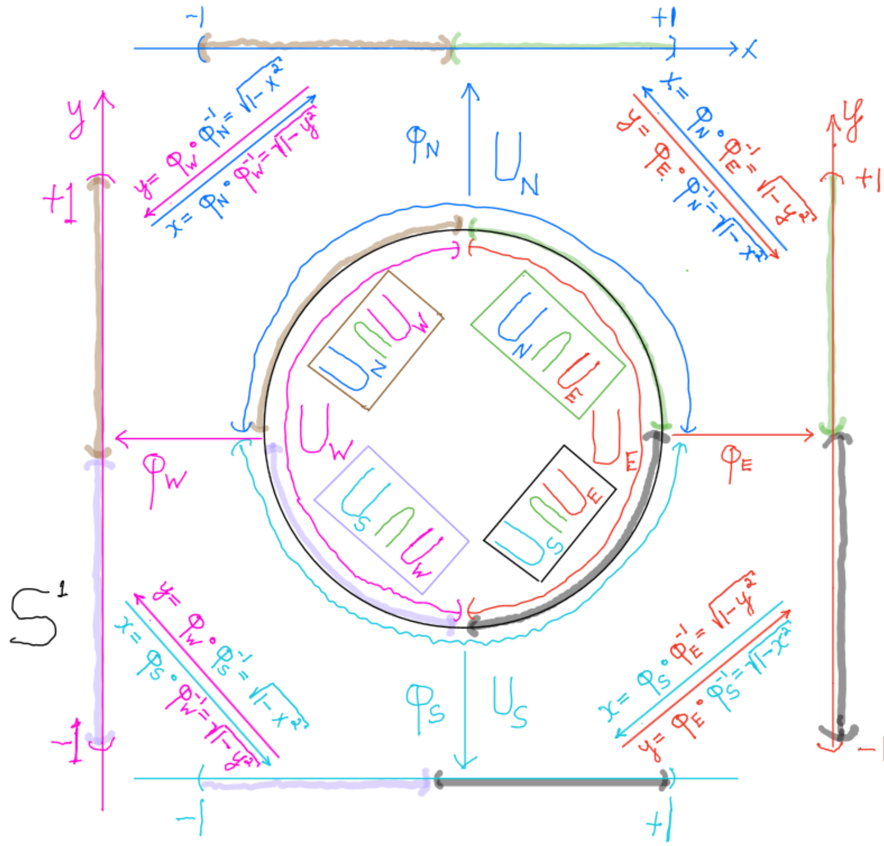


Figure 21.1a: Manifold $M=S^1$, two coordinate charts (φ_N, U_N) and (φ_E, U_E), the transition map $x=\varphi_N\circ\varphi_E^{-1}=\sqrt{1-y^2}$, and its inverse $y=\varphi_E\circ\varphi_N^{-1}=\sqrt{1-x^2}$



$U_N = \{\theta : 0<\theta<\pi\}$	$\varphi_N(\theta) = \cos\theta$
$U_W = \{\theta : \frac{\pi}{2}<\theta<\frac{3\pi}{2}\}$	$\varphi_W(\theta) = \sin\theta$
$U_S = \{\theta : -\pi<\theta<0\}$	$\varphi_S(\theta) = \cos\theta$
$U_E = \{\theta : -\frac{\pi}{2}<\theta<\frac{\pi}{2}\}$	$\varphi_E(\theta) = \sin\theta$

Figure 21.1b: The boxed four coordinate charts together with their transition functions $\varphi_N \circ \varphi_E^{-1}, \varphi_E^{-1} \circ \varphi_N, \varphi_E \circ \varphi_S^{-1}, \varphi_S \circ \varphi_E^{-1}, \cdots$ form a particular atlas, say $\mathcal{A}_1$, on the unit sphere S^1. Because its transition function are infinitely differentiable, i.e. C^∞, one says that this C^∞ atlas is smooth. If $\mathcal{A}_2$ is another smooth atlas, then one says that $\mathcal{A}_1$ and $\mathcal{A}_2$ are smoothly equivalent provided their union $\mathcal{A}_1 \cup \mathcal{A}_2$ is also a smooth atlas. One says that $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_1 \cup \mathcal{A}_2$ are smooth structures on S^1. Thereby S^1 is a smooth manifold. Being smoothly equivalent these atlases form an equivalence class of smooth structures. By taking the union of of all atlases belonging to a smooth structure one obtain a maximal smooth atlas.

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The hallmark of a manifold is that it serves as the foundational domain – the base or the arena – that accommodates different types of physical or geometrical structures: scalar fields, vector fields and tensor fields as they arise from multilinear algebra.
Because of this feature one thinks and speaks of the manifold as the base manifold of the system.

Experimentally and observationally, physicists, scientist and engineers have found that the laws that rule the universe are independent of coordinate system which the observer uses to acquire the data on which these laws are based.

A base manifold and the various physical or geometrical structures above it reflect this fact.

Let f be a real-valued function defined on an n-dimensional manifold M.

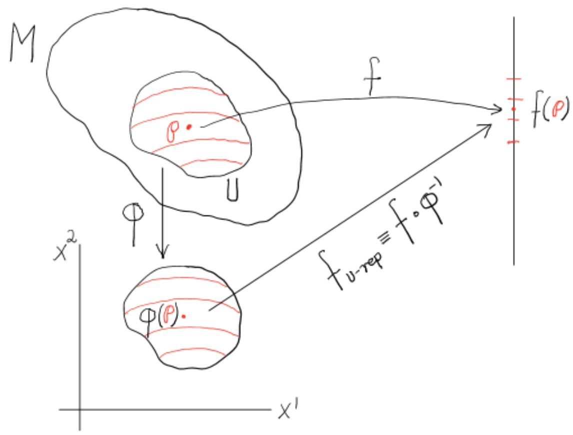

Figure 21.2: Real valued function f and its coordinate representative $f\circ\varphi^{-1} \equiv f_{U\text{-rep}}$ relative to the coordinate chart (φ, U). **21.5**

The function

$$f\circ\varphi^{-1}(x^1,\dots,x^n) \equiv f_{U\text{-rep}}(x^1,\dots,x^n)$$

is the $\underline{\varphi\text{-representative}}$ of f:

a) $f\circ\varphi^{-1}$ is a real valued function whose domain is $\varphi(U)$

$$f\circ\varphi^{-1} : \begin{cases} \varphi(U) \longrightarrow f\circ\varphi^{-1}(\varphi(U)) = f(U) \\ \left(x^1(P),\dots,x^n(P)\right) = \varphi(P) \rightsquigarrow f\circ\varphi^{-1}\left(x^1(P),\dots,x^n(P)\right) = f\circ\varphi^{-1}(\varphi(P)) = f(P) \end{cases}$$

b) If $f\circ\varphi^{-1}$ is C^∞ at $\varphi(P)\in\mathbb{R}^n$, one says that f is smooth, i.e. is C^∞ at $P\in M$.

c) The differentiability of f at P is independent of the compatible charts containing P.

Indeed let (φ_U, U) and (φ_V, V) be two overlapping charts containing P so that $U\cap V \neq \emptyset$.

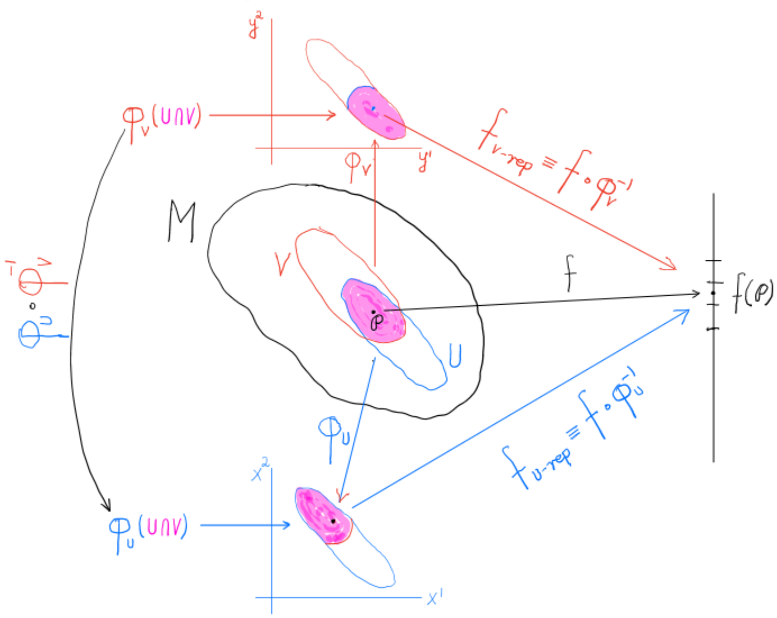

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Figure 21.3: The scalar function f is represented by $f_{U\text{-rep}} = f \circ \varphi_U^{-1}$ relative (φ_U, U) and by $f_{V\text{-rep}} = f \circ \varphi_V^{-1}$ relative to (φ_V, V). However, one has

$$\underbrace{f \circ \varphi_U^{-1}(x^1, \dots, x^n)}_{\substack{\text{III} \\ f_{U\text{-rep}}(x^1, \dots, x^n)}} = \underbrace{f \circ \varphi_V^{-1}(y^1, \dots, y^n)}_{\substack{\text{III} \\ f_{V\text{-rep}}(y^1, \dots, y^n)}}$$

The two coordinate representatives are related by the composition with the transition map and it inverse

$$\boxed{\begin{aligned} f \circ \varphi_V^{-1} &= \left(f \circ \varphi_U^{-1}\right) \circ \left(\varphi_U \circ \varphi_V^{-1}\right) \\ f \circ \varphi_U^{-1} &= \left(f \circ \varphi_V^{-1}\right) \circ \left(\varphi_V \circ \varphi_U^{-1}\right) \end{aligned}}$$

Since $\varphi_U \circ \varphi_V^{-1}(y^1, \dots, y^n)$ is smooth, both $f \circ \varphi_V^{-1}(y^1, \dots, y^n)$ and $f \circ \varphi_U^{-1}(x^1, \dots, x^n)$ are differentiable with respect to y^i and x^δ respectively.

Example 1.

Consider $\boxed{M = S^1 = \{(x, y) : x = \cos\theta; y = \sin\theta; -\pi < \theta \leq \pi\}}$ with its two overlapping neighborhoods:

$$U_U = U_N = \{(x, y) : x = \cos\theta; y = \sin\theta; \quad 0 < \theta < \pi\}$$

$$U_V = U_E = \{(x, y) : x = \cos\theta; y = \sin\theta; -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\}$$

$$U_N \cap U_E = \{(x, y) : x = \cos\theta; y = \sin\theta; \quad 0 < \theta < \tfrac{\pi}{2}\}$$

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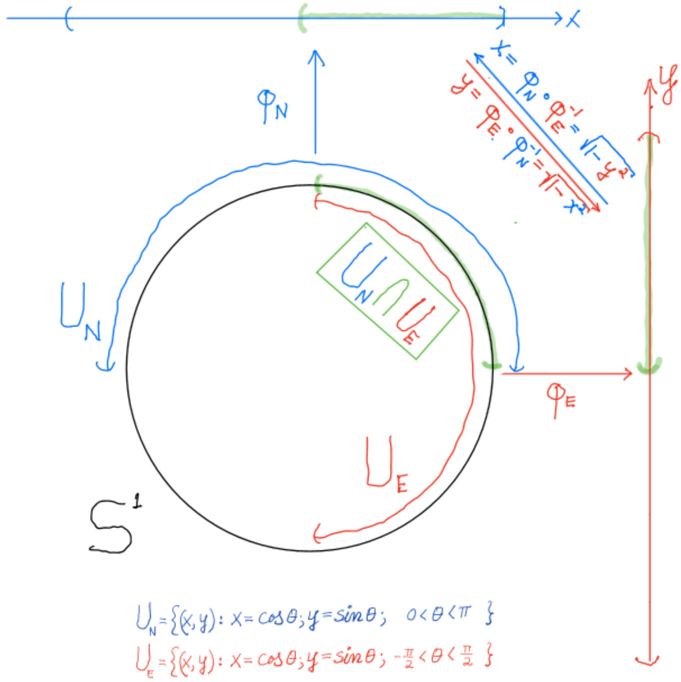

Figure 21.4: $M = S^1$ as the domain for the scalar field $f = \sin 2\theta$.

Let $f = \sin 2\theta = 2\sin\theta\cos\theta = 2yx$

then
$$f_{N\text{-rep}}(x) = f \circ \varphi_N^{-1}(x)$$
$$= 2\sin(\cos^{-1}x)\cos(\cos^{-1}x)$$
$$= 2\sqrt{1-x^2}\, x$$

and
$$f_{E\text{-rep}}(y) = f \circ \varphi_E^{-1}(y)$$
$$= 2\sin(\sin^{-1}y)\cos(\sin^{-1}y)$$
$$= 2y\sqrt{1-y^2}$$

The transition maps are
$$x = x = \varphi_N \circ \varphi_E^{-1}(y) = \sqrt{1-y^2}; \quad 0 < y < 1$$
$$y = y = \varphi_E \circ \varphi_N^{-1}(x) = \sqrt{1-x^2}; \quad 0 < x < 1$$

These transition maps transform $f_{N\text{-rep}}$ into $f_{E\text{-rep}}$ and vice versa as follows:

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$$\begin{aligned} f_{N\text{-rep}}(x) &\equiv f\circ\varphi_N^{-1}(x) \\ &= f\circ\varphi_N^{-1}\left(\varphi_N\circ\varphi_E^{-1}(y)\right) \\ &= f\circ\left(\varphi_N^{-1}\circ\varphi_N\right)\circ\varphi_E^{-1}(y) \\ &= f\circ\varphi_E(y) \\ &\equiv f_{E\text{-rep}}(y), \end{aligned}$$

and similarly for $f_{E\text{-rep}}$ into $f_{N\text{-rep}}$

III. Transition map as the link between overlapping charts

There are two essential ingredients in forming a manifold as a valid concept: (i) the existence of coordinate charts that cover the manifold, and (ii) the transition map ("transformation," i.e. consistency) between them when they overlap.

Example 2 (Real $n\times n$ non-singular matrices)

Let $M=\{A\}=\{[a_{ij}]\}=$ set of all $n\times n$ real non-singular matrices.
Let $\Delta: R^{n^2}\to R$ be the determinant function: $\Delta=\det[a_{ij}]$.
Let $\Delta^{-1}(R^1-\{0\})=$ the set of all matrices having non-zero determinant.
M is a manifold.
(i) The chart (φ,U) with

$$\varphi:\quad U=M\longrightarrow\varphi(U)\qquad(\text{is open in }R^{n^2})$$
$$A\rightsquigarrow\varphi(A)=\left(\varphi_{11}(A)=a_{11},\varphi_{12}(A)=a_{12},\cdots,\varphi_{nn}(A)=a_{nn}\right)$$

is a globally defined coordinate chart ("system"): $\varphi(U)=R^{n^2}$

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(ii) The chart $(\bar{\varphi}, \bar{U})$ with

$$\bar{\varphi}: \quad \bar{U} = M \longrightarrow \bar{\varphi}(\bar{U}) \qquad (\text{is open in } R^{n^2})$$

$$A \rightsquigarrow \bar{O}(A) = \left(\bar{\varphi}_{11}(A) = a_{12}, \bar{\varphi}_{12}(A) = a_{11}, \cdots, \varphi_{nn}(A) = a_{nn}\right)$$

is another globally defined coordinate system.

(iii) These two charts are C^∞ related because the <u>transition</u> <u>map</u>

$$\bar{\varphi}\circ\bar{\varphi}^{-1}: \qquad R^{n^2} \longrightarrow R^{n^2}$$

$$(a_{11}, a_{12}, \cdots, a_{nn}) \rightsquigarrow \bar{\varphi}\circ\varphi^{-1}(a_{11}, a_{12}, \cdots, a_{nn}) = (a_{12}, a_{11}, \cdots, a_{nn})$$

is infinitely differentiable. Indeed, its Jacobian, the matrix of its partial derivatives, is

$$J^i{}_j(\bar{\varphi}\circ\varphi^{-1}) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Summary

Transition maps, together with their Jacobian derivatives, are the means for transforming geometrical structures (scalar fields, vector fields, etc) between overlapping coordinate charts.

APPENDIX: The two coordinate representatives in Problem 1 of Homework 0. (21.10)

In homework 0, problem 1 dealt with two representatives of the same function f, a solution to the Helmholtz equation $(\nabla^2+k^2)f=0$,

$$f\circ\varphi^{-1}(x,y)=\psi(x,y) \text{ and}$$

$$f\circ\bar\varphi^{-1}(\bar x,\bar y)=\bar\psi(\bar x,\bar y).$$

The transition functions between the two coordinate charts ("coordinate systems") (U,φ) and $(\bar U,\bar\varphi)$ were

$$\varphi\circ\bar\varphi^{-1}(\bar x,\bar y):\begin{cases} x= \cos\theta\,\bar x+\sin\theta\,\bar y \\ y=-\sin\theta\,\bar x+\cos\theta\,\bar y\end{cases} \qquad (A.1)$$

Its inverse is

$$\bar\varphi^{-1}\circ\varphi(x,y): \begin{matrix}\bar x=\cos\theta\, x-\sin\theta\, y \\ \bar y=\sin\theta\, x+\cos\theta\, y\end{matrix} \qquad (A.2)$$

In the context of that problem, a specific solution f to $(\nabla^2+k^2)f=0$ had two coordinate representatives.

Relative to the chart (U,φ): $f_{U\,rep}(x,y)=f\circ\varphi^{-1}(x,y)\ (=\psi(x,y))$

Relative to the chart $(\bar U,\bar\varphi)$: $f_{\bar U\,rep}(\bar x,\bar y)=f\circ\bar\varphi^{-1}(\bar x,\bar y)\ (=\bar\psi(\bar x,\bar y))$

Example (Plane wave solution)

Consider the plane wave solution $f(\vec x)=e^{i\vec k\cdot\vec x}$ where $\vec k\cdot\vec k=k^2$ in $(\nabla^2+k^2)f=0$.

Then (i) $f_{U\,rep}(x,y)=f\circ\varphi^{-1}(x,y)=e^{i(k_x x+k_y y)}=\psi(x,y)$ where $k_x^2+k_y^2=k^2$ and $\varphi(\vec x)=(x,y)$ is the (U,φ) representative of f

and (ii) $f_{\bar U\,rep}(\bar x,\bar y)=f\circ\bar\varphi(\bar x,\bar y)=e^{i(\bar k_x\bar x+\bar k_y\bar y)}=\bar\psi(\bar x,\bar y)$ where $\bar k_x^2+\bar k_y^2=k^2$ and $\bar\varphi(\vec x)=(\bar x,\bar y)$ is the $(\bar U,\bar\varphi)$ representative of f

Nota bene: If $(\bar x,\bar y)$ is related to (x,y) by a rotation (="transition function"), then $(\bar k_x,\bar k_y)$ is necessarily related to (k_x,k_y) by the inverse rotation (= "inverse of the rotation function")