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LECTURE 22

I. Tangent Vectors: Overview

II. Tangent Vector: Its Definition

III. Tangent Vector as a Coordinate Invariant Concept

Singer and Thorpe: Chapter 5: p 99-100

Comment: In their definition of a tangent vector on page 99 their wording is in terms of "then there exists an n-tuple $(a_1, a_2, \cdots, a_n)$ of ···" The red underlined "exists" is an unfortunate choice of bad terminology. With a reader's background in linear algebra, this term suggests that the existence of that n-tuple is a matter of proving the existence and uniqueness of a system of n linear equation for its n unknowns $a_1, a_2, \cdots, a_n$. This, however, is not all the case. Instead, these a_i's come about as the components of the tangent (velocity) of a given curve. In fact, Singer and Thorpe say/admit precisely that at the top of their page 104 in the context of their Figure 5.2.

Read Section 1.3 ("Vectors and Vector Fields") in Notes on Differential Geometry by Noel J. Hicks

22.2

I. Tangent vectors: Overview

A tangent vector is a mathematical method for geometrizing the concept of change in terms of a directional derivative so as to guarantee that it reflects the physical requiremt of being invariant under ("independent of") one's choice of coordinates.

Given an n-dimensional manifold M, the coordinate charts of M accommodate the existence of a vector space

$$V = T_P(M),$$

the tangent space $T_P(M)$ at each point P of M. Each element of $T_P(M)$ is a tangent vector, an adaptation of the familiar direction derivative to the landscape (i.e. the coordinate charts) surrounding $P \in M$.

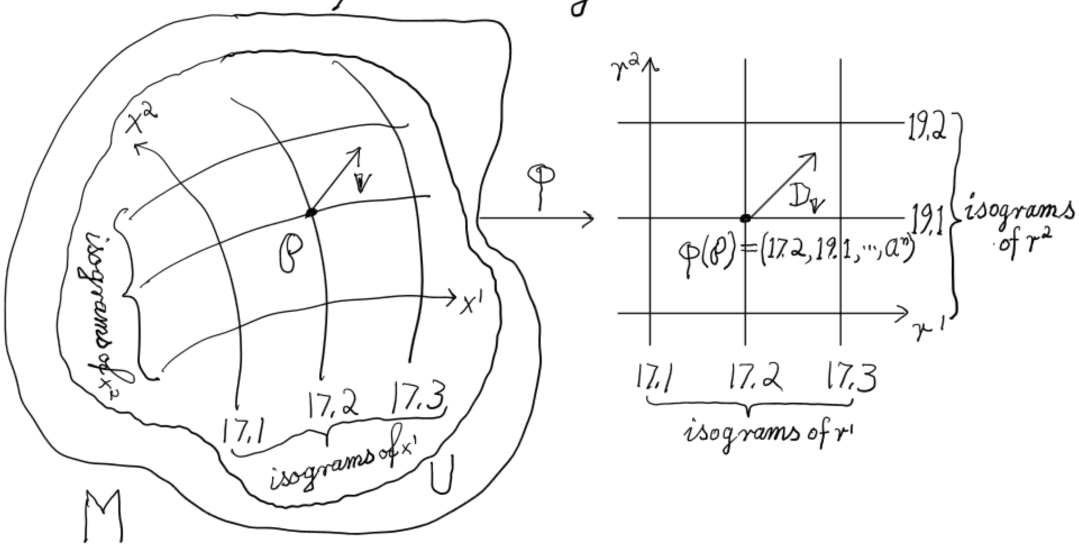

Figure 22.1: Vector v coordinatized as the directional derivative D_v relative to the coordinate chart (φ, U). The point P is coordinatized by the components $x^i(P) \equiv (\varphi(P))^i \equiv r^i \circ \varphi(P)$, $i = 1, \cdots n$. One calls $r^i : R^n \to R$, $(a^1, \cdots, a^n) \rightsquigarrow r^i(a^1, \cdots, a^n) = a^i$ the i^{th} coordinate function on R^n, while $x^i \equiv (\varphi)^i \equiv r^i \circ \varphi$ is the i^{th} coordinate function on M.

Thus,

(i) a tangent vector at a point is a directional derivative that sends all functions smooth around that point into the reals, and

(ii) that tangent vector is determined by its values on all smooth functions.

This overview is concretized by the following three-step construction of a vector at a point P in a given manifold M.

STEP 1

First define the concept of the coordinate functions on R^n and M.

Definition (Coordinate function)

a) Let M be an n-dimensional manifold.

b) Consider the chart (φ, U).

c) Consider the i^{th} coordinate function on R^n, namely,

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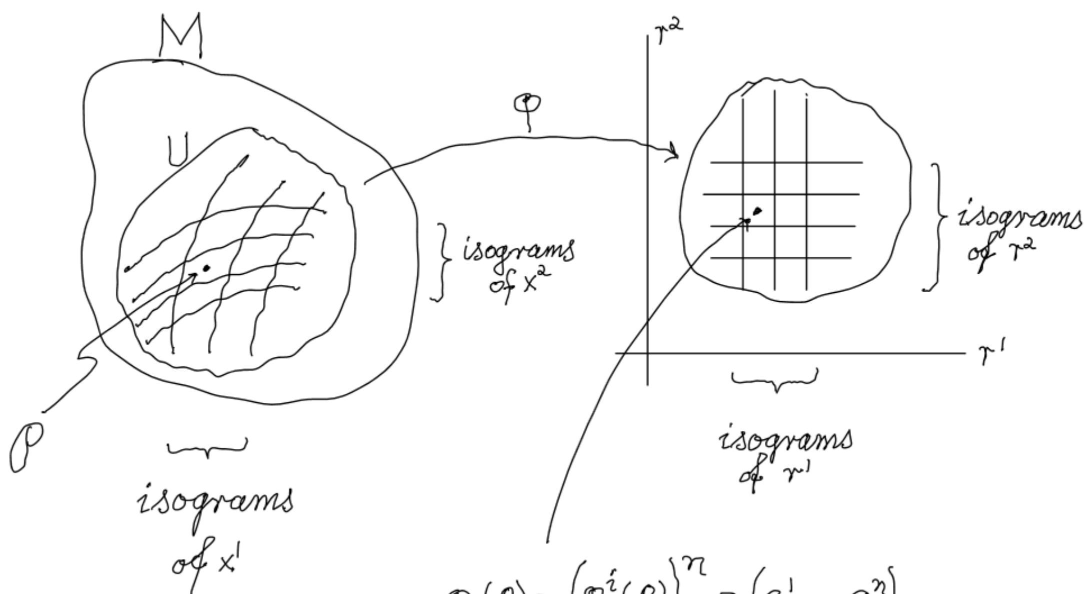
$$r^i: \quad \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$(a^1, \dots, a^n) \rightsquigarrow r^i(a^1, \dots, a^n) = a^i \qquad (22.1)$$

(Comment.
In computer science and in some text books, e.g. "Notes on Differential Geometry" by N.J. Hicks, this is called the i^{th} slot function. It picks out the i^{th} element of $(a^1, \dots, a^n) \in \mathbb{R}^n$.)

Then

$$x^i = r^i \circ \varphi_U \left(= (\varphi_U)^i\right) : \quad U \longrightarrow \mathbb{R}$$
$$p \rightsquigarrow x^i(p) = r^i \circ \varphi_U(p) \qquad (22.2)$$

is the i^{th} coordinate function of (φ_U, U) on U



$$\varphi_U(p) = \left(\varphi^i(p)\right)_{i=1}^{n} = (a^1, \dots, a^n)$$

Figure 22.2: Contours (="isograms") of the coordinate functions of (φ_U, U) on U and on $\mathbb{R}^n$.

STEP 2 (22.6)

Definition (Tangent vector as a derivation)

Let $P \in M$ be a point in the manifold M.
A tangent vector v at P is the map

$$v: \quad C^\infty(M, P, R) \longrightarrow R$$

$$\begin{pmatrix} \text{smooth real} \\ \text{functions in a} \\ \text{nbhd of } P \in M \end{pmatrix} \longrightarrow \text{Reals}$$

$$f \rightsquigarrow v(f) \in R$$

with the following properties: If $P \in U$, the domain of the chart (φ_U, U), then there exists $(\alpha^1, \dots, \alpha^n) \in R^n$ such that

$$\boxed{v(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f \circ \varphi_U^{-1}(r^1, \dots, r^n)\Big|_{\varphi(P)=(a^1,\dots,a^n)}} \qquad (22.3a)$$

$$= \sum_{i=1}^{n} \alpha^i D_i f_{U\text{-rep}}(r^1, \dots, r^n)\Big|_{(a^1,\dots,a^n)} \qquad (22.3b)$$

Comment

1. Equations (22.3) are valid for any differentiable function f. Thus, the coefficients $\{\alpha^1, \dots, \alpha^n\}$ determine the mapping v in a unique way.
2. Conversely, given the mapping v, the coefficients $\{\alpha^1, \dots, \alpha^n\}$ are uniquely determined. Indeed, consider the independent set of n differentiable functions

$$f^1_{\varphi\text{-rep}}(r_1, \dots, r^n) = r^1$$

$$f^2_{\varphi\text{-rep}}(r_1, \dots, r^n) = r^2$$

$$\vdots$$

$$f^{n}_{\varphi\text{-rep}}(r_1,\dots,r^n) = r^n$$

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These function are determined implicitly by (φ_0, U). Indeed, they are the coordinate functions defined on R^n as exhibited by Eq. (22.1) on page 22.4.

Apply Eqs (22.3) to these functions,

$$\mathbb{V}(f^1) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f^1_{\varphi\text{-rep}}(r^1,\dots,r^n)\Big|_{(a^1,\dots,a^n)} = \alpha^1$$

$$\mathbb{V}(f^2) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f^2_{\varphi\text{-rep}}(r^1,\dots,r^n)\Big|_{(a^1,\dots,a^n)} = \alpha^2$$

$$\vdots$$

$$\mathbb{V}(f^n) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f^n_{\varphi\text{-rep}}(r^1,\dots,r^n)\Big|_{(a^1,\dots,a^n)} = \alpha^n$$

Thus the vector $\mathbb{V}$ evaluated on any scalar function f is

$$\mathbb{V}(f) = \sum_{i=1}^{n} \mathbb{V}(f^i) \frac{\partial}{\partial r^i} f_{\varphi\text{-rep}}(r^1,\dots,r^n)\Big|_{(a^1,\dots,a^n)} \qquad (22.4)$$

Equation (22.4) is the $\{\alpha^i = \mathbb{V}(f^i)\}$ linear combination of the partial derivatives of $f_{U\text{-rep}}$ at the point $(a^1,\dots,a^n)$, the coordinate image in R^n of point P. Thus there is a one-to-one correspondence between $\mathbb{V}$ and the given n-tuple $(\alpha^1,\dots,\alpha^n) = (\mathbb{V}(f^1),\dots,\mathbb{V}(f^n))$

$$\boxed{\mathbb{V} \longleftrightarrow (\alpha^1,\dots,\alpha^n) \in R^n}\,.$$

However, this observation lacks objectivity [lookup "objectivity" in the Ayn Rand Lexicon]. This is because it is subjective: it depends solely on the observer: on the chosen coordinate chart.

STEP 3 (22.8)

In order to get reality right, one must show that Eqs (22.3) also hold for those charts which have a non-zero intersection with (φ_U, U) given in the definition.

In other words, is Eq. (22.3a) invariant under coordinate transformations?

The tangent vector $\mathbb{V} : C^\infty(M, R, \mathcal{P}) \longrightarrow R^1$ at point $\mathcal{P}$ has as its defining property the formula

$$\mathbb{V}(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f \circ \varphi_U^{-1}(r^1, \ldots, r^n)\Big|_{\varphi_U(\mathcal{P}) = (a^1, \ldots, a^n)}$$

relative to chart (φ_U, U).

Insert the identity map $\varphi_{\bar{U}}^{-1} \circ \varphi_{\bar{U}}$ of the overlapping chart $(\bar{\varphi}_{\bar{U}}, \bar{U})$ to obtain

$$\mathbb{V}(f)\Big|_{\mathcal{P}} = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f \circ \varphi_U^{-1}(r^1, \ldots, r^n)\Big|_{\varphi_U(\mathcal{P})}$$

$$= \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} \underbrace{f \circ \bar{\varphi}_{\bar{U}}^{-1}}_{f_{\bar{U}\text{-rep}}(\bar{r}^1, \ldots, \bar{r}^n)} \circ \underbrace{\bar{\varphi}_{\bar{U}} \circ \varphi_U^{-1}(r^1, \ldots, r^n)}_{\{\bar{r}^j(r^1, \ldots, r^n)\}_{j=1}^{n}}\Big|_{\varphi_U(\mathcal{P})}$$

This insertion has resulted in the $\bar{U}$-representative of f,

$$f_{\bar{U}\text{-rep}} = f \circ \varphi_{\bar{U}}^{-1}$$

being composed with the transition function $\varphi_{\bar{U}} \circ \varphi_U^{-1}$. Consequently, the partial derivatives $\frac{\partial}{\partial r^i}$ are those of a composite function with intermediate variables $\bar{r}^1, \ldots, \bar{r}^n$.

Explicitly one has

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$$V(f)=\sum_{i=1}^{n}\alpha^i\frac{\partial}{\partial r^i}\overbrace{f_{\bar{U}\text{-rep}}\left(\bar{g}^1(r^1,\dots,r^n),\dots,\bar{g}^n(r^1,\dots,r^n)\right)}^{f_{\bar{U}\text{-rep}}\circ\bar{\varphi}_{\bar{U}}\circ\varphi_U^{-1}(r^1,\dots,r^n)}.$$

where $\quad \bar{g}^j(r^1,\dots,r^n)=\bar{r}^j\circ\bar{\varphi}_{\bar{U}}\circ\varphi_U^{-1}(r^1,\dots,r^n) \qquad j=1,\dots,n$

are components of the transition function $\bar{\varphi}_{\bar{U}}\circ\varphi_U^{-1}(r^1,\dots,r^n)$.

Apply the chain rule to find each partial derivative of the "outer" function $f_{\bar{U}\text{-rep}}(\bar{r}^1,\dots,\bar{r}^n)$ with its multicomponent "inner" function

$$\left\{\bar{g}^j(r^1,\dots,r^n)=\bar{r}^j\circ\bar{\varphi}_{\bar{U}}\circ\varphi_U^{-1}(r^1,\dots,r^n)\right\}_{j=1}^{n}$$

The result is

$$V(f)=\sum_{i=1}^{n}\alpha^i\sum_{j=1}^{n}\left.\frac{\partial\bar{r}^j}{\partial r^i}\right|_{\varphi_U(P)=(a^i)_{i=1}^n}\times\left.\frac{\partial}{\partial\bar{r}^j}f_{\bar{U}\text{-rep}}(\bar{r}^1,\dots,\bar{r}^n)\right|_{\bar{\varphi}_{\bar{U}}(P)=\left(g^j(a^1,\dots,a^n)\right)_{j=1}^n}$$

Here the partial derivatives of the components of the transition function $\bar{\varphi}_{\bar{U}}\circ\varphi_U^{-1}$,

$$\frac{\partial\bar{r}^j}{\partial r^i}(r^1,\dots,r^n)=\frac{\partial}{\partial r^i}\,\bar{r}^j\circ\bar{\varphi}_{\bar{U}}\circ\varphi_U^{-1}(r^1,\dots,r^n), \qquad (22.5)$$

are the elements of the Jacobian matrix

$$\left[\frac{\partial\bar{r}^j}{\partial r^i}\right]\equiv\left[J^j{}_i(\bar{\varphi}_{\bar{U}}\circ\varphi_U^{-1})\right] \qquad (\text{"Jacobian"})$$

It transforms the φ_U coordinate components $\{\alpha^i\}$ to

$$\boxed{\sum_{i=1}^{n}\alpha^i\frac{\partial\bar{r}^j}{\partial r^i}\equiv\beta^j} \qquad j=1,\dots,n,$$

of the new $\bar{\varphi}_{\bar{U}}$ coordinate system.

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Relative this system the defining relation for $\mathbb{v}$ has the form

$$\mathbb{v}(f) = \sum_{j=1}^{n} \beta^j \frac{\partial f_{\bar{U}\text{-rep}}(\bar{r}^1, \dots, \bar{r}^n)}{\partial \bar{r}^j}\bigg|_{\bar{\varphi}_{\bar{U}}(P)}$$

$$= \sum_{j=1} \beta^j \frac{\partial}{\partial \bar{r}^j} f \circ \bar{\varphi}_{\bar{U}}^{-1}\bigg|_{\bar{\varphi}_{\bar{U}}(P) = (\bar{a}^1, \dots, \bar{a}^n)}, \qquad (22.6)$$

which is the same relative to (φ_U, U),

$$\mathbb{v}(f) = \sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i} f \circ \varphi_U^{-1}\bigg|_{\varphi_U(P) = (a^1, \dots, a^n)} \qquad (22.7)$$

Because of (22.6) and (22.7) one says that $\mathbb{v}(f)$ is a <u>coordinate invariant</u>

The preceding 3-step construction of the concept "vector" is condensed into the following definition:

In the context of manifold-based multivariable calculus, a vector at a point is a linear combination of partial derivatives whose expansion coefficients are related to those of any other expansion by means of the Jacobian (matrix) of the transition map between their respective overlapping coordinate charts.

The explicit structure of the above definition is highlighted in its expanded version as follows:

The genus of a vector v at P consists of the collection of linear combinations of partial derivatives

$$\sum_{i=1}^{n} \alpha^i \frac{\partial}{\partial r^i}\,,\; \sum_{j=1}^{n} \beta^j \frac{\partial}{\partial \bar{r}^j}\,,\; \sum_{k=1}^{n} \bar{\bar{\alpha}}^k \frac{\partial}{\partial \bar{\bar{r}}^k}\,,\; \cdots \qquad (22.12)$$

in their respective overlapping coordinate charts

$$(\varphi_U, U)\,,\; (\varphi_{\bar U}, \bar U)\,,\; (\varphi_{\bar{\bar U}}, \bar{\bar U})\,,\; \cdots .$$

The differentia of a vector v at P is the fact that the expansion coefficients of one be related to those of another by means of the Jacobian matrix of the transition function between the respective charts.

The principle of unit-economy [see "unit-economy" in the Ayn Rand Lexicon] demands that this form equivalence be reflected in the notation for the vector v. This is achieved by recalling from Figure 22.2 that on the neighborhood of the coordinate chart (φ_U, U) its components are the coordinate functions

$$x^i : \begin{array}{l} U \longrightarrow \mathbb{R} \\ P \rightsquigarrow x^i(P) \equiv r^i \circ \varphi_U(P) \equiv (\varphi_U(P))^i \end{array} \qquad i = 1, \cdots, n$$

In light of this, the unit-economical notation for $\frac{\partial}{\partial r^i}(f \circ \varphi_U^{-1})\Big|_{\varphi_U(P)}$ in Eq. (22.7) is

$$\frac{\partial(f)}{\partial x^i} \equiv \frac{\partial}{\partial r^i}(f \circ \varphi_U^{-1})\Big|_{\varphi_U(P)} = \frac{\partial}{\partial r^i} f_{U\text{-rep}}(r^1, \cdots, r^n) \qquad (22.8)$$

for $f \in C^\infty(M, P, \mathbb{R})$. Thus, $\frac{\partial}{\partial x^i}$ corresponds relative to the coordinate system φ_U to the n-tuple $(\alpha^i)_{i=1}^{n} = (0, \cdots, 1, \cdots, 0)$, where its i^{th} entry is 1.

Comment (22.13)

The primary purpose of the introduction of the coordinate functions x^i by means of the formula

$$x^i = r^i \circ \varphi \qquad i = 1, \dots, n \qquad (22.9)$$

is, of course, to mathematize the result of a cartographer drawing a coordinate grid on the neighbor hood U of the manifold, in other words, to make explicit the measurements,

$$x^i(\mathcal{P}) = r^i \circ \varphi(\mathcal{P}) \qquad i = 1, \dots, n \ , \qquad (22.10)$$

necessary to identify a metaphysiclly given $\mathcal{P}$ in the neighborhood U.

However, there is also a psychological/epistemological purpose. Equation (22.8) directs readers to their responsibility of taking cognizance of the fact there is always the φ_U-defined domain in R^n where the calculation are performed.

These two purposes also apply to the definition of a vector v in terms of the partial derivatives Eq. (22.8), namely

$$\left.\frac{\partial(f)}{\partial x^i}\right|_{\mathcal{P}} \equiv \left.\frac{\partial}{\partial r^i} f \circ \varphi_U^{-1}(r^1, \dots, r^n)\right|_{\varphi_U(\mathcal{P})} \qquad i = 1, \dots, n \ . \qquad (22.11)$$

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More generally one has

$$\mathbb{V}(f)=\sum_{i=1}^{n}\alpha^i\frac{\partial}{\partial x^i}(f),$$

or by leaving f as-yet-unspecified,

$$\boxed{\mathbb{V}=\alpha^i\frac{\partial}{\partial x^i}} \qquad (22.8)$$

Thus $\mathbb{V}$ is the rate of change of an as-yet-unspecified function into the direction of the n-tuple $(\alpha^1,\dots,\alpha^n)$ relative to the coordinates φ.

In a similar way, based on the coordinate functions y^j, one obtains

$$\frac{\partial(f)}{\partial y^j}\equiv\frac{\partial}{\partial\bar{r}^j}\left(f\circ\psi_V^{-1}\right)\Big|_{\psi_V(p)}=\frac{\partial}{\partial\bar{r}^j}f_{V\text{-rep}}(\bar{r}^1,\dots,\bar{r}^n)$$

and

$$\boxed{\mathbb{V}=\beta^j\frac{\partial}{\partial y^j}} \qquad (22.9)$$

Combining Eqs. (22.8)-(22.9) obtain

$$\alpha^i\frac{\partial}{\partial x^i}=\mathbb{V}=\beta^j\frac{\partial}{\partial y^j},$$

which is to say that a vector at a point is a coordinate independent object.

If x^i and y^j are the coordinate functions on $U\cap V$, then reference to Eq. (22.2) yields

$$\frac{\partial}{\partial x^i}=\frac{\partial(y^j)}{\partial x^i}\frac{\partial}{\partial y^j}$$

where $\frac{\partial(y^j)}{\partial x^i}$ are the elements of the Jacobian matrix.