LECTURE 23

23.1

I. Tangent Vector

II. Tangent Vector as a derivation

III. The Tangent Space $T_p(M)$

IV. The Cotangent Space $T_p^*(M)$

V. Coordinate Transformation

Read Singer & Thorpe's
"Lecture Notes on Elementary Geometry and Topology",
pages 100-101

Read Section 1.3 ("Vectors and Vector Fields") in
<u>Notes on Differential Geometry</u> by Noel J. Hicks

I. Tangent Vector

(23.2)

The concept of a tangent vector is the kingpin by which one grasps multi-variable calculus and linear algebra from a single pespective. It is the the portal from linear to non-linear mathematics.

It maps C^∞ scalar functions on a manifold neighborhood U surrounding a point P into the reals

$$
\begin{aligned}
v : C^\infty(M,P,R) &\longrightarrow R \\
f &\rightsquigarrow v(f) \equiv \alpha^i \frac{\partial}{\partial x^i}\left(f\circ\varphi_U^{-1}\right)\Big|_{\varphi_U(P)} \\
&\qquad = \alpha^i \frac{\partial}{\partial r^i} f_{U\text{-rep}}(r^1,\dots,r^n)\Big|_{\varphi_U(P)}
\end{aligned}
$$

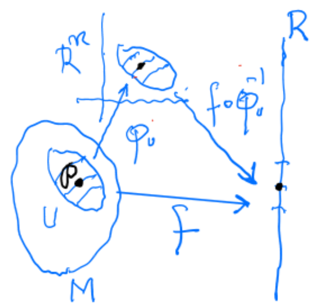

Figure 23.1: Scalar function f as a superstructure rooted in a base-manifold M, the foundation for all geometrical superstructures.

If $x^j = r^j \circ \varphi_U$, $j = 1,\dots,n$ are φ_U's coordinate functions, then by the principle of unit-economy [according to which one reduces a vast amount of information (here, that of the manifold substructure) to a minimal number of units] one writes

$$v(f) = \alpha^i \frac{\partial(f)}{\partial x^i} \qquad (23.1)$$

for all functions smooth at P. Thus one arrives at the conclusion that

$$\boxed{v = \alpha^i \frac{\partial}{\partial x^i}} \qquad (23.2)$$

Comment

1) This is a mapping that sends smooth functions into the values of their

directional derivative $D_v = \alpha^i \frac{\partial}{\partial x^i}$ evaluated at the point P. (23.3)

2) The distinguishing feature of the set of tangent vectors v at P is their one-to-one correspondence with n-tuples

$$v \rightsquigarrow (\alpha^1, \dots, \alpha^n) \in R^n$$

at P relative to the coordinate chart (φ_U, U).

II. Tangent Vector as the Derivation at a Point.

The concept "tangent vector v" subsumes and integrates a non-trivial multiplicity of diverse components, attributes, and environments. Nevertheless, all of them can be condensed into a triad of defining statements:

For any $f, g \in C^\infty(M, P, R)$ and for $\lambda \in R$

$$\boxed{\begin{aligned} &(1)\quad v(f+g) = v(f) + v(g) \\ &(2)\quad v(\lambda f) = \lambda v(f) \\ &(3)\quad v(fg) = f v(g) + v(f) g \end{aligned}}$$

(1), (2) "linearity"

(3) "the product rule"

A mapping which satifies these three properties is called a derivation. In addition, starting with a mapping having these properties, one can show that relative to any coordinate system, say (φ_U, U) containing P, there exist $(\alpha^1, \dots, \alpha^n) \in R^n$

such that

$$\boxed{v = \alpha^i \frac{\partial}{\partial x^i}},$$

where x^i is the i^{th} coordinate function of φ_U.

Conclusion

A tangent vector is a derivation, and a derivation is a a tangent vector.

III. The Tangent Space $T_P(M)$ at $P \in M$ (23.4)

a) The fact that numbers can be added and multiplied by scalars so as to result in new numbers can by extended to the set $T_P(M)$ of tangent vectors at point P in M.
In that case one says that the set $T_P(M)$ of tangent vectors at P is a vector space: it is closed under addition and scalar multiplication of its elements:

$$(i)\quad (\mathbb{V}_1 + \mathbb{V}_2)(f) = \mathbb{V}_1(f) + \mathbb{V}_2(f) \qquad \mathbb{V}_1, \mathbb{V}_2 \in T_P(M);\ \forall f \in C^\infty(M,P,R)$$

$$(ii)\quad (\lambda \mathbb{V})(f) = \lambda \mathbb{V}(f) \qquad \mathbb{V} \in T_P(M);\ \forall f \in C^\infty(M,P,R)$$

b) The vector space $T_P(M)$ is isomorphic to R^n. Indeed,

$$\begin{aligned} R^n &\longleftrightarrow T_P(M) \\ (\alpha^1, \dots, \alpha^n) &\rightsquigarrow \alpha^i \frac{\partial}{\partial x^i} \end{aligned}$$

is an isomorphism induced by the coordinate chart (φ_U, U).

c) The chart (φ_U, U) causes $T_P(M)$ to have the <u>natural coordinate basis</u>

$$\left\{ \delta_1^i \frac{\partial}{\partial x^i}, \dots, \delta_n^i \frac{\partial}{\partial x^i} \right\} = \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\} \equiv \{ e_1, \dots, e_n \}$$

Thus
$$\boxed{T_P(M) = \operatorname{span}\left(\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1}^n \right)}$$

d) The transition from one coordinate chart to another induces a corresponding change in bases from $\{ \frac{\partial}{\partial x^i} \}$ to $\{ \frac{\partial}{\partial y^j} \}$. They are related by

$$\frac{\partial}{\partial x^i} = \frac{\partial (y^j)}{\partial x^j} \frac{\partial}{\partial y^j},$$

where

$$\frac{\partial (y^j)}{\partial x^i} = J^j_{\ i} \left(\psi_V \circ \varphi_U^{-1} \right) \equiv \frac{\partial \mathcal{T}^j}{\partial r^i}(r^1, \dots, r^n) \Big|_{\varphi(P)}$$

are the matrix elements of the Jacobian matrix between the two coordinate systems (φ_U, U) and (ψ_V, V).

23.5

Example ("Trivial Tangent Space")

Consider $M = R^n$ and $P = a \in R^n$. Then the tangent space $T_a(R^n)$ at the point $a \in R^n$ is natrally isomorphic to R^n itself. The isomorphism

$$R^n \longleftrightarrow T_a(R^n)$$

is given by

$$(\lambda^1, \dots, \lambda^n) \longleftrightarrow \lambda^i \frac{\partial}{\partial r^i}\ .$$

IV. The Cotangent Space $T_P^*(M)$ dual to $T_P(M)$

Consider a scalar function $f(x^k)$, its isograms in the neighborhood of a point P, and its Taylor series around that point.

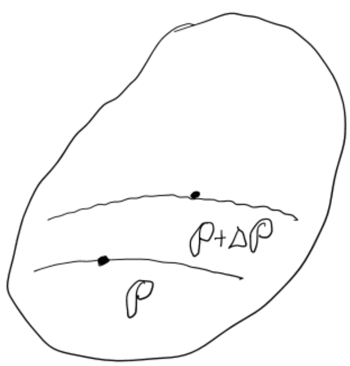

Figure 23. Two points, P and its neighbor $P+\Delta P$, situated on two neighboring isograms of the function f.

23.6

$$\Delta f = f\left(x^k(\mathcal{P})+\Delta x^k\right) - f\left(x^k(\mathcal{P})\right) = f\circ\varphi(\mathcal{P}+\Delta\mathcal{P}) - f\circ\varphi(\mathcal{P})$$
$$= \cancel{f\left(x^k(\mathcal{P})\right)} + \frac{\partial f}{\partial x^i}\Big|_{x^k(\mathcal{P})}\Delta x^i + \frac{1}{2!}\frac{\partial^2 f}{\partial x^i \partial x^j}\Big|_{x^k(\mathcal{P})}\Delta x^i \Delta x^j + \cdots - \cancel{f\left(x^k(\mathcal{P})\right)}$$
$$= \frac{\partial f}{\partial x^i}\,\delta^i_k\,\Delta x^k + \cdots$$
$$= \frac{\partial f}{\partial x^i}\langle \omega^i | e_k\rangle \Delta x^k + \cdots$$
$$= \left\langle \frac{\partial f}{\partial x^i}\omega^i \,\Big|\, \Delta x^k e_k \right\rangle + \cdots$$

where $\mathrm{span}\left(\left\{e_k = \frac{\partial}{\partial x^k}\right\}_{k=1}^n\right) = T_{\mathcal{P}}(M)$ and $\mathrm{span}\left(\{\omega^i\}_{i=1}^n\right) = T^*_{\mathcal{P}}(M)$

By setting $\omega^i = dx^i$ so that

$$\delta^i_k = \langle \omega^i | e_k\rangle = \left\langle dx^i \Big| \frac{\partial}{\partial x^k}\right\rangle = \frac{\partial x^i}{\partial x^k},$$

one has the fact that $\{dx^i\}_{i=1}^n$ is a (linearly independent) spanning set for the space dual to $T_{\mathcal{P}}(M)$, namely the cotangent space $T^*_{\mathcal{P}}(M)$:

$$\mathrm{span}\left(\{dx^i\}_{i=1}^n\right) = T^*_{\mathcal{P}}(M)$$

Consequently, with these tangent and cotangent bases, one has

$$\Delta f = \left\langle \frac{\partial f}{\partial x^i}dx^i \,\Big|\, \Delta x^k \frac{\partial}{\partial x^k}\right\rangle + \cdots$$

The linear function (a.k.a. covector) $\frac{\partial f}{\partial x^i}dx^i = df \in T^*_{\mathcal{P}}(M)$ is called the differential of f.

Thus, evaluating df on the displacement vector $\Delta\mathcal{P} \equiv \Delta x^k \frac{\partial}{\partial x^k}$ yields the principal linear part of Δf,

$$\boxed{\begin{aligned}\Delta f &= \left\langle df \,\Big|\, \Delta x^k \frac{\partial}{\partial x^k}\right\rangle + \cdots \\ &= \frac{\partial f}{\partial x^k}\Delta x^k + \cdots\end{aligned}}$$

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As a result of Eqs. (23.1)-(23.2), each function f in the domain $C^\infty(M, P, R)$ of

$$\mathbb{V}: f \rightsquigarrow \mathbb{V}(f)\Big|_P = \alpha^i \frac{\partial f}{\partial x^i}$$

determines a unique linear function on $T_P(M)$. It is given by the

Definition ("The differential of f")

$$df: \quad T_P(M) \longrightarrow R$$

$$\mathbb{V} \quad \rightsquigarrow \langle df|\mathbb{V}\rangle\Big|_P = \mathbb{V}(f)\Big|_P$$

$$\rightsquigarrow \langle df|\widehat{\alpha^i \frac{\partial}{\partial x^i}}\rangle = \alpha^i \frac{\partial f}{\partial x^i}$$

By virtue of this definition df is linear on $T_P(M)$. This is because

(i) $\langle df \mid \mathbb{V}_1+\mathbb{V}_2\rangle = (\mathbb{V}_1+\mathbb{V}_2)(f)$

$$= \mathbb{V}_1(f)+\mathbb{V}_2(f)$$

$$= \langle df|\mathbb{V}_1\rangle + \langle df|\mathbb{V}_2\rangle$$

and

(ii) $\langle df|\lambda\mathbb{V}\rangle = (\lambda\mathbb{V})(f)$

$$= \lambda\,\mathbb{V}(f)$$

$$= \lambda\langle df|\mathbb{V}\rangle .$$

23.8

The linear function df on $T_P(M)$ is called the differential (or the differential form) of f at the point P.

a) The set of differential forms at point P, $T_P^*(M)$, is a vector space. Indeed, it is closed under addition and scalar multiplication:

$$
\begin{aligned}
(i)\quad \langle df+dg|\mathbb{V}\rangle &\equiv \langle df|\mathbb{V}\rangle + \langle dg|\mathbb{V}\rangle \\
&= \mathbb{V}(f)+\mathbb{V}(g) \qquad \left(=\alpha^i \frac{\partial f\circ\varphi^{-1}(r^1,\dots,r^n)}{\partial r^i} + \alpha^i\frac{\partial g\circ\varphi^{-1}(r^1,\dots,r^n)}{\partial r^i}\right) \\
&= \mathbb{V}(f+g) \\
&= \langle d(f+g)|\mathbb{V}\rangle.
\end{aligned}
$$

Thus $df+dg$ is the differential of $f+g$ at P and therefore belongs to $T_P^*(M)$. Similarly, one has

$$
\begin{aligned}
(ii)\quad \langle \lambda df|\mathbb{V}\rangle &\equiv \lambda\langle df|\mathbb{V}\rangle \\
&= \lambda\,\mathbb{V}(f) \\
&= \mathbb{V}(\lambda f) \\
&= \langle d(\lambda f)|\mathbb{V}\rangle
\end{aligned}
$$

Thus $\lambda df = d(\lambda f)$ is the differential of λf at P. Consequently, $T_P^*(M)$ is a vector space indeed.

b) Every given coordinate system induces its natural coordinate basis for $T_P(M)$. The corresponding dual basis is given by the following

Proposition ("Basis Dual to a Coordinate Basis")

Let $\{e_i \equiv \frac{\partial}{\partial x^i}\}_{i=1}^{n}$ be the natural coordinate basis for $T_p(M)$ (23.9)
induced by (φ_U, U).
Let $\{x^j\}_{j=1}^{n}$ be the coordinate functions of φ
Conclusion:

$$\langle dx^j | \tfrac{\partial}{\partial x^i} \rangle = \delta^j_{\ i}\,.$$

i.e. $\{dx^j\}$ is the basis dual to $\{e_i = \frac{\partial}{\partial x^i}\}$ at P.
By letting in turn

$$V = \frac{\partial}{\partial x^i} \qquad i = 1, \dots, n$$

$$f = x^j \qquad j = 1, \dots, n$$

and computing $V(f)$ one finds using $\langle df | V \rangle = V(f) = \alpha^i \frac{\partial f \circ \varphi(r^1, \dots, r^n)}{\partial r^i}$ that

$$\langle dx^j | \tfrac{\partial}{\partial x^i} \rangle = \frac{\partial x^j \circ \varphi(r^1, \dots, r^n)}{\partial r^i}$$

$$\shortparallel \quad = \frac{\partial r^j}{\partial r^i}$$

$$\boxed{\langle dx^j | \tfrac{\partial}{\partial x^i} \rangle = \delta^j_{\ i}}$$

One says that $\{dx^j\}$ is the natural coordinate basis for $T_p^*(M)$ induced by (φ_U, U).

Thus

$$\boxed{T_p^*(M) = \text{span}\left(\{dx^j\}_{j=1}^{n}\right)}$$

This to be compared and contrasted with

$$\boxed{T_p(M) = \text{span}\left(\{\tfrac{\partial}{\partial x^i}\}_{i=1}^{n}\right)}$$

and

$$\left\{\frac{\partial}{\partial x^i}\right\} \longleftrightarrow \{dx^j\}$$

V. Coordinate Transformation

(23.10)

Let (φ_U, U) and (ψ_V, V) be two coordinate charts surrounding P.

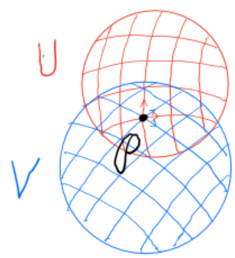
On $U \cap V$ one has two bases and their duals:

Both satisfy their duality relations at P:

$$\langle dx^j | \frac{\partial}{\partial x^i} \rangle = \delta^j_i \quad \text{and} \quad \langle dy^\ell | \frac{\partial}{\partial y^k} \rangle = \delta^\ell_k .$$

From page 22.8 one knows that the two vector bases are related by

$$\frac{\partial}{\partial x^i} = \frac{\partial (y^k)}{\partial x^i} \frac{\partial}{\partial y^k} .$$



$$\frac{\partial}{\partial x^i} = \frac{\partial (y^j)}{\partial x^i} \frac{\partial}{\partial y^j}$$

Figure 23.2: Relation between basis at P relative to U and that relative to V.

What does this relation imply about the relationship between the corresponding dual bases?

A: In $T^*_P(M)$ consider the covector

$$\frac{\partial (x^j)}{\partial y^\ell} dy^\ell .$$

Its value on $\frac{\partial}{\partial x^i}$ is

$$\begin{aligned} \langle \frac{\partial (x^j)}{\partial y^\ell} dy^\ell | \frac{\partial}{\partial x^i} \rangle &= \frac{\partial (x^j)}{\partial y^\ell} \langle dy^\ell | \frac{\partial}{\partial x^i} \rangle \\ &= \frac{\partial (x^j)}{\partial y^\ell} \frac{\partial (y^\ell)}{\partial x^i} \end{aligned}$$

Using the chain rule one obtains

$$\begin{aligned} &= \frac{\partial (x^j)}{\partial x^i} \\ &= \delta^j_i . \end{aligned}$$

Compare this equality with

$$\langle dx^j | \frac{\partial}{\partial x^i} \rangle = \delta^j_i$$

23.11

and conclude that

$$\boxed{\frac{\partial (x^j)}{\partial y^\ell} dy^\ell = dx^j}$$

This is because

"Two linear functions are equal if and only if they have the same value when evaluated on a set of basis vectors."