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LECTURE 25

Vector Field as a Flow

I. Theorems: Where do they come from?

II. Disassociated tangent spaces

III. Vector field as a flow field

IV. Flow field as a transformation

V. Flow field as a transformation group

For a clear summary of Lectures 22-24 ((i) where do vectors come from? (ii) what are they? how does one mathematize them? (iii) what happens under a change of coordinates? (iv) what is the differential of a function?) read Sections 9.2-9.4 in MTW

Read Section 1.5 ("Curves and Integral Curves") in
Notes on Differential Geometry by Noel J. Hicks

Alternatively,
read P125-127 in Chap. 5 of Singer and Thorpe's "Lecture Notes on Elementary Topology and Geometry", which is developed in Lecture 25.

I. Theorems: Where do they come from? or: Is Deduction Possible without Induction?

Many admirers of mathematics, be they philosophers, engineers, physicists, including mathematicians themselves, hold mathematics in high regard as a deductive science. However, they are missing the point. Before one has deduction one must have induction. Before one deduces that Socrates is a man, one must have induced (ultimately starting from observation) that all men are mortal.

Mathematical theorems are a good example. Before one proves a theorem (like the one on page 25.4), i.e. before one deduces certain consequences from what is known, given, or assumed, one must assemble, identify precisely, and state explicitly the starting point from which the subsequent consequences follow.

This assemblying, identification, and articulation process – the inductive process – is much more difficult than a deduction. This is because induction is based on <u>all</u> of one's knowledge, a person's <u>whole</u> cognitive landscape.

Thus, because of limitations of time, resources, or even ignorance, perseverance and intelligence, instructors quite often skimp on the motivation, i.e. the necessary inducitive process which gives rise to the generalization expressed by the theorem, theory, principle, etc. This is done with the hope and understaning that the student's curiosity,, will pick up where the instructor did (or had to) slack off.

II. Disassociated Tangent Spaces

Every point of a differentiable manifold M accommodates a tangent space which consists of vectors, i.e.

a) of elements that correspond to n-tuples in R^n, or

b) of derivatives, or

c) of tangents to curves in the manifold.

Thus, at each point of a diffentiable manifold M there is a vector space space, the tangent space $T_p(M)$ at p.
However, there is as yet no relation between all the vectors in $T_p(M)$ and all those in $T_{p'}(M)$ at different points p and p'

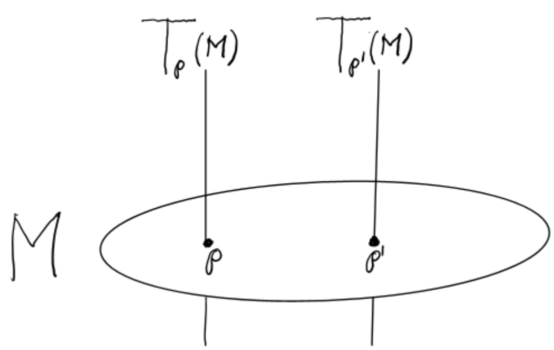

Figure 25.1: Tangent spaces $T_p(M)$ and $T_{p'}(M)$ are disjoint and are as yet unrelated.

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Even with a given vector field u, i.e. an assignment of a vector u_P to each $T_P(M)$, there is no correspondence between $T_P(M)$ and $T_{P'}(M)$.
However, a vector field on M is a structure which does establish a correspondence between points on M, but not between the tangent spaces of a manifold.

III. Vector Field as a Flow

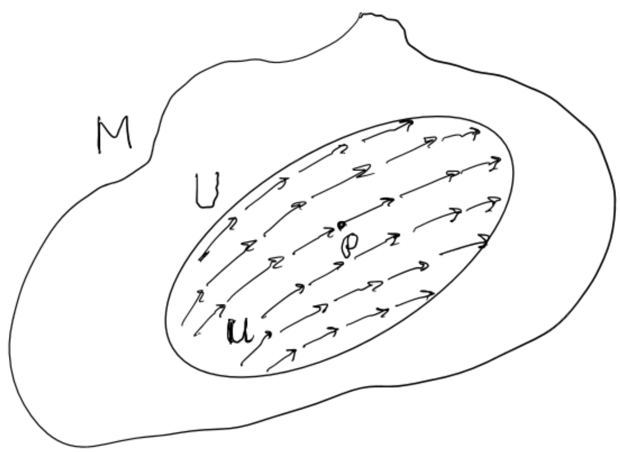

Figure 25.2: Vector field u in the neighborhood U of a point P.

A vector field induces a correspondence between points in a manifold. It is mathematized by the integral curves of the vector field viewed as a flow field. The existence of these curves is highlighted by the following

Definition ("Integral curve of a vector field")

Given: A smooth vector field (Figure 25.2) 25.5

$$u = u^i(x^1, \cdots, x^n)\frac{\partial}{\partial x^i}$$

on M.

An <u>integral curve</u> of u is a smooth curve

$$c(\tau) : (a, b) \longrightarrow M$$

such that the vector $\dot{c}(\tau)$ tangent to c at each point $P \in c$ is the vector $u(P)$ of the field u that has been assigned to $T_P(M)$:

$$\dot{c}(\tau) = u(c(\tau)) \qquad a < \tau < b$$

or

$$\frac{dc^i}{d\tau}\frac{\partial}{\partial x^i} = u^i(c^1(\tau), \dots, c^n(\tau))\frac{\partial}{\partial x^i}$$

or

$$\frac{dc^i}{d\tau} = u^i(c^1(\tau), \dots, c^n(\tau)) \qquad i = 1, \cdots, n$$

This is a system of n equations in n to-be-determined functions $c^i(\tau)$. One says that the integral curve <u>fits</u> the given vector field.

The existence and uniqueness of integral curves, i.e. of solutions to the system of 1^{st} order o.d.e.'s is a feature of dynamical systems, including Hamiltonian systems, fluid dynamics, geodesic flows, in fact all systems of 1^{st} order o.d.e.'s. Their features are captured by the following

Theorem ("Existence and Uniqueness of a Flow Field")

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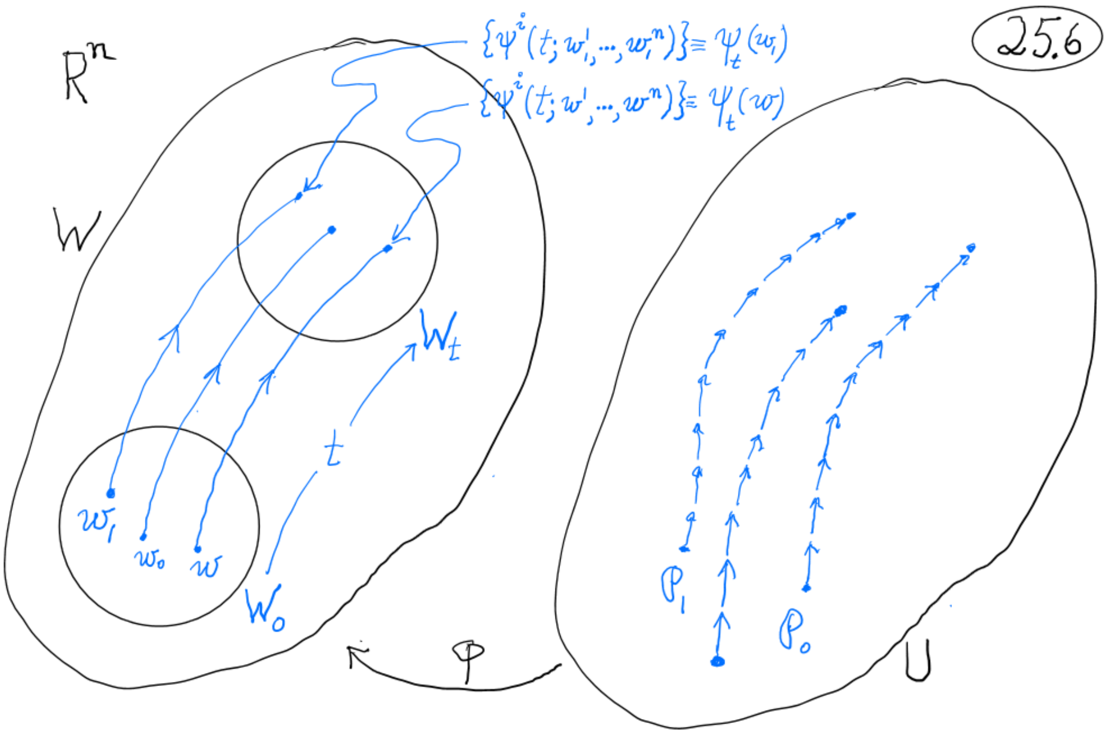

Figure 25.3: Integral curves of a vector field via their coordinate representation (φ, U).

Given: Let W be an open set R^n

Let $w_0 = (w_0^1, \dots, w_0^n) \in R^n$

Let $u^i(x^1, \dots, x^n) \in C^\infty(W, R)$, $i = 1, \dots, n$, a set of functions smooth on W

(Nota bene: They form the φ-representative of the vector field $\mathbf{u} = u^i \frac{\partial}{\partial x^i}$)

Conclusion:

There exists

(i) an open set $W_0 \subset W$ about w_0

(ii) an open interval $(-\epsilon, \epsilon) \subset R$

(iii) a unique smooth n-component map

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$$\psi=\{\psi^i\}_{i=1}^n : (-\epsilon,\epsilon)\times W_0 \longrightarrow W$$

$$(t;w^1,\dots,w^n) \rightsquigarrow \psi(t;w^1,\dots,w^n)=\{\psi^i(t;w^1,\dots,w^n)\} \quad (25.1)$$

such that ψ is a solution to the equation

$$\frac{d\,c^i(t)}{dt}=u^i(c^1(t),\dots,c^n(t)) \qquad i=1,\dots,n$$

subject to

$$c(0)=w\,,\ \text{i.e.}\ \ c^i(0)=w^i\,,\ \ i=1,\dots,n$$

Remark

The w-parametrized family of solutions, Eq. (25.1) implies the flow field

$$\psi(t;w)\equiv\psi_t(w)$$

induced by the vector field u. Indeed, by shifting attention from the evolution parameter t to the n-component initial values $w=\{w^i\}$ of the curve $c_w(t)$, one arrives at

$$\{c_w^i(t)\}=\{\psi^i(t;w)\}\equiv\{\psi_t^i(w)\}$$

with the property that

$$w=\{c_w^i(0)\}=\psi_{t=0}(w).$$

The flow field $\psi_t(w)$ is said to be induced (or "generated") by the vector field u because it satisfies the system of differential equations

$$\frac{d}{dt}c_w^i(t)=u^i(c_w^1(t),\dots,c_w^n(t)),$$

or equivalently

$$\boxed{\frac{d}{dt}\psi_t(w)=u(\psi_t(w))}$$

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IV. Flow Field as a Transformation

If $C^i_w(t) = \psi^i(t;w)$ for $i=1,\ldots,n$, why does one use different symbols to refer to one and the same thing?

The reason is that the u-determined flow has two different aspects. The symbol $\{C^i_w\}$ is the reminder that the flow consists of individual curves that connect different points. By contrast, the notation

$$\{\psi^i(t;w)\} \equiv \Psi_t(w)$$

emphasizes the fact that Ψ_t is a transformation that acts on the whole neighborhood W_0 and transforms it into the new neighborhood $W_t = \Psi_t(W_0)$ at a different location without referring to the specific curves along which this happens.

(25.9)

V. Flow Transformations form a Group under Composition

Recall that the flow of a vector field u is a $w = \{w^1, \ldots, w^n\}$ parametrized family of curves $c_w(t) \equiv \psi_t(w)$ with tangents that are the preexisting vectors of the vector field.
The existence and uniqueness Theorem on pages 25.6 – 25.7 implies that every family member of the flow has the same additive property, namely

$$\psi_\tau \circ \psi_{\tau'}(w) = \psi_{\tau+\tau'}(w).$$

This property is brought to light by the notational shift in the Remark on page 25.7. There the flow fuction ψ_t was introduced by the definition

$$c_w(t) = \{\psi^i(t; w^1, \ldots, w^n)\} \equiv \psi_t(w) \qquad (25.2)$$

Then, as depicted in Figure 25.4,

$$\psi_t : \quad W_0 \longrightarrow W_t = \psi_t(W_0) \qquad (25.3)$$
$$w \rightsquigarrow \psi_t(w)$$

is a t-parametrized group of one-to-one transformations that act on sets of points in the manifold.

a) Let $t = \tau' + \tau$ and apply the flow Theorem on pages 25.5 – 25.6 to the initial value data in W_0. The resulting transformation depicted in Figure 25.4a is

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$$\psi_{\tau'+\tau} : W_0 \longrightarrow \psi_{\tau'+\tau}(W_0) \equiv W_{\tau'+\tau} \qquad (25.4)$$

$$w \rightsquigarrow w'' = \psi_{\tau'+\tau}(w) \in W_{\tau'+\tau} \qquad (25.5)$$

This transformation exist and is unique for all $-\epsilon < \tau', \tau, \tau'+\tau < 0$.
In particular, for $\tau = 0$ note that

$$W_{\tau'+0} = \psi_{\tau'+0}(W_0) \qquad (25.6)$$

$$w' = \psi_{\tau'+0}(w). \qquad (25.7)$$

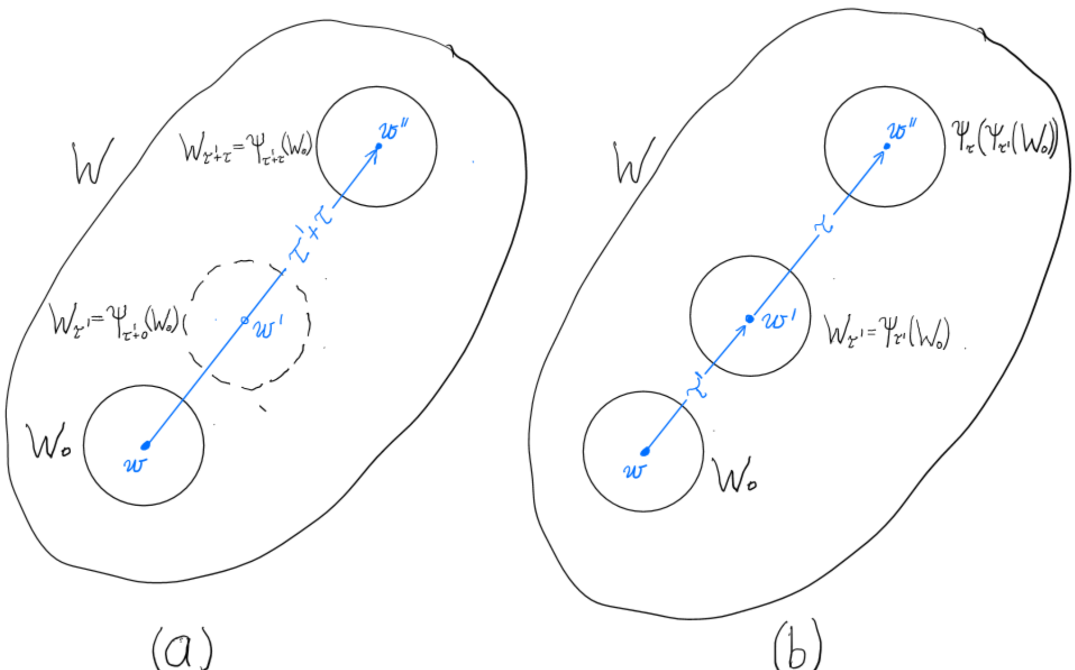

Figure 25.4: The u-induced transformation ψ_t applied to W_0 in panel (a) results in $\psi_{\tau'+\tau}(W_0)$. In panel (b) it results in $\psi_\tau \circ \psi_{\tau'}(W_0)$. But both of them are also the transform images of one and the same $\psi_{\tau'}(W_0)$, the neighborhood centered around w'! Because of uniqueness the two are identical, i.e. $\psi_{\tau'+\tau}(W_0) = \psi_\tau \circ \psi_{\tau'}(W_0)$.

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b) Let $t=\tau'$ and apply the flow Theorem on pages 25.6–25.7 to the initial value data in W_0. The resulting transformation depicted in Figure 25.4b is

$$\psi_{\tau'}: W_0 \longrightarrow \psi_{\tau'}(W_0) \equiv W_{\tau'} \qquad (25.8)$$
$$w \rightsquigarrow w' = \psi_{\tau'}(w) \in W_{\tau'} \qquad (25.9)$$

Using the points in the transformed neighborhood $W_{\tau'}$, Eq. (25.7), as initial value data, reapply to it the flow Theorem but with $t=\tau$. The resulting transformation depicted in Figure 25.4b is

$$\psi_\tau: \quad \psi_{\tau'}(W_0) \longrightarrow \psi_\tau(\psi_{\tau'}(W_0)) = \psi_\tau \circ \psi_{\tau'}(W_0) \qquad (25.10)$$
$$w' = \psi_{\tau'}(w) \rightsquigarrow w'' = \psi_\tau(\psi_{\tau'}(w)) = \psi_\tau \circ \psi_{\tau'}(w) \qquad (25.11)$$

c) Compare Eqs. (25.10)–(25.11) with Eqs. (25.4)–(25.5) and infer that for all w in W_0 one has

$$\psi_{\tau+\tau'}(w) = \psi_\tau \circ \psi_{\tau'}(w). \qquad (25.12)$$

Indeed, the l.h.s is according to Eq. (25.5) the image of $\psi_{\tau'}(w)$, Eq. (25.7), and the r.h.s. is according to Eq. (25.11) the image of the same $\psi_{\tau'}(w)$. Thus, because of the uniqueness of these images, the two sides are equal for all $w \in W_0$:

$$\boxed{\psi_{\tau+\tau'} = \psi_\tau \circ \psi_{\tau'}} \qquad (25.13)$$

whose domain is (25.12)

$$\psi^{-1}_{\tau'}(W_{\tau'}) = W_0 = \psi^{-1}_{\tau+\tau'}(W_{\tau''})$$

whenever $-\epsilon < \tau, \tau', \tau+\tau' < \epsilon$.

d) By letting $\tau = -\tau'$ in the boxed Eq. (25.13) obtain

$$\psi_0(w) = \psi_{-\tau'} \circ \psi_{\tau'}(w). \qquad (25.14)$$

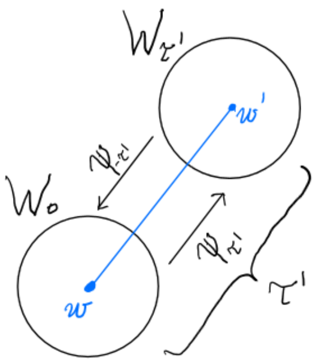

Figure 25.5: Flow transformation $\psi_{-\tau'}$ composed with $\psi_{\tau'}$ yields $\psi_{-\tau'} \circ \psi_{\tau'} = \psi_0$, the identity transformation. Hence $\psi_{-\tau'} = \psi^{-1}_{\tau'}$.

But from the defining Eqs (25.2)-(25.3) and by using Eq.(25.1) in the flow Theorem on page 25.7, one has

$$\psi_0(w) = w \qquad (25.15)$$

for all $w \in W_0$. Consequently, with Eq.(25.7) obtain

$$w = \psi_{-\tau'} \circ \psi_{\tau'}(w) \qquad \forall w \in W_0$$

Thus,

$$\psi_{-\tau'} \circ \psi_{\tau'} = \text{identity}$$

and

$$\boxed{\psi_{-\tau'} = \psi^{-1}_{\tau'}} \qquad (25.16)$$

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Conclusion

The two boxed Eqs. (25.13) and (25.16)

$$\boxed{\psi_{\tau+\tau'} = \psi_\tau \circ \psi_{\tau'}}$$

$$\boxed{\psi_{-\tau'} = \psi_{\tau'}^{-1}}$$

imply that the ψ's form a local 1-parameter (τ) group of transformations.