LECTURE C26

Complement:

1. Integral curves: Point tranformations
2. Functional transformation: Lie transport of a scalar field
3. Lie derivative of a scalar field
4. Lie transport of a vector field
5. Lie derivative of a vector field

In MTW read Box 8.4, Sect 9.2, 9.6

In Singer & Thorpe read p126 (or P142 in the Springer edition)

(C26,2)

In mathematics the importance of a concept can be gauged by the number of contexts where it plays a central role. A vector u at P, because of its multi-faced nature is, such a concept. It is

(i) an element in $T_P(M)$ Lecture 23

(ii) a derivation at P Lecture 24

(iii) the tangent to a curve at P Lecture 25

(iv) a vectorial displacement generator Lecture 26

(v) the exponent in the exponential map $e^{\tau u}$ Lecture 26

Recap I: Integral Curves of a Vector Field.

GIVEN: (i) a vector field $u = u^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i}$, and hence the <u>autonomous</u> system of differential equations

$$\frac{dc^i(t)}{dt} = u^i(c^1(t), \ldots, c^n(t)) \quad i = 1, \ldots, n \qquad (C26.1)$$

(ii) an initial point $w \equiv \{w^i\}_{i=1}^n$

CONCLUSION: There <u>exists</u> a <u>unique</u> integral curve c which <u>fits</u> u and passes through w, i.e.

$$c: (-\varepsilon, \varepsilon) \longrightarrow R^n$$

$$t \rightsquigarrow c(t) = \{c^i(t; w^1, \ldots, w^n)\}_{i=1}^n$$

such that (i) $c(t)$ is a solution to Eqs. (26.1) and

(ii) $\{c^i(t=0; w^1, \ldots, w^n) = w^i\}_{i=1}^n = w$

C26.3

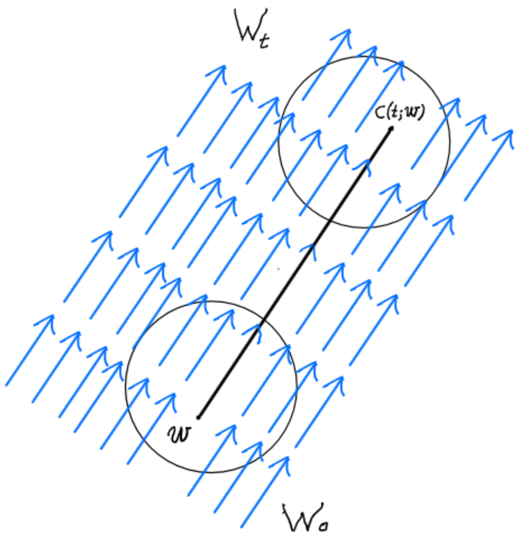

Figure 26.1 A curve $c(t;w)$ which fits the given vector field u

I. Integral curves as Transformations

1. Point Transformations

The existence of integral curves $c(t;w)$ with distinct starting points w implies the existence of a point transformation

$$\begin{aligned} \psi_t : \quad W_o \subset R^n &\longrightarrow W_t \subset R^n \\ w \quad &\rightsquigarrow \psi_t(w) = c(t;w) \end{aligned}$$

where ψ_t satisfies the given system of differential equations

$$\frac{d\,\psi_t^i(w)}{dt} \frac{\partial}{\partial x^i} = u^i(\psi_t(w)) \frac{\partial}{\partial x^i} \,,$$

or more abstractly,

C26.4

$$\frac{d\psi_t}{dt} = \mathfrak{u}(\psi_t).$$

This transformation has two properties:

(a) closure under composition, i.e. the composite of two transformations is another point transformation:

$$\psi_{\tau_2}(\psi_{\tau_1}(w)) \equiv \boxed{\psi_{\tau_2} \circ \psi_{\tau_1}(w) = \psi_{\tau_2+\tau_1}(w)} \quad \forall\, w \in W_0$$

(b) the inverse of a transformation is also a point transformation:

$$\boxed{\psi_t^{-1}(w) = \psi_{-t}(w)} \quad \forall\, w \in W_0$$

2. Functional Transformations

The existence of integral curves determines a functional transformation

$$\psi_t^* : C^\infty(M, w, R) \longrightarrow C^\infty(M, w, R)$$

$$f \rightsquigarrow \psi_t^*(f) = g$$

on the space of smooth functions. The new function g is obtain from the old function f by the requirement that the value of the new function at the new point equals the value of the old function at the old point:

$$\underbrace{\psi_t^*}_{\substack{\text{new function}\\(g)}}\Big(f\big(\underbrace{\psi_t(w_0)}_{\text{new point}}\big)\Big) = \underbrace{f}_{\substack{\text{old}\\\text{function}}}(\underbrace{w_0}_{\text{old point}}),$$

C26.5

or in terms of the newly defined function g

$$g \circ \Psi_t = f. \qquad (26.2)$$

For good reasons f is often called the "pullback of g by Ψ_t."

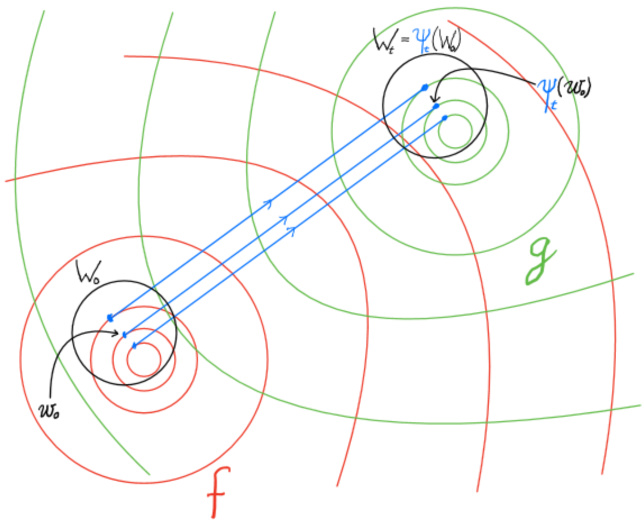

Figure 26.2 The point transformation Ψ_t maps not only points such as $w_0 \in W_0$ to $\Psi_t(w_0) \in \Psi_t(W_0)$ but also their red isograms of the function f onto the corresponding green isograms of the function g.
The relation between the two functions is

$$g(w) = f(\Psi_t^{-1}(w)), \text{ i.e., } g = f \circ \Psi_t^{-1}.$$

The function g is called the pullback of f by Ψ_t^{-1}.
The function $f = g \circ \Psi_t$ is called the pullback of g by Ψ_t.

It follows that

$$\Psi_t^*(f(w)) = f(\Psi_t^{-1}(w)) \equiv g(w) \quad \forall w \in M,$$

which is to say that the u-induced transformation on $C^\infty(M,R)$ is

$$\Psi_t^* : C^\infty(M,R) \longrightarrow C^\infty(M,R)$$
$$f \rightsquigarrow \Psi_t^*(f) = f(\Psi_t^{-1})$$

C 26.6

Example: Transformation induced by translations on R^1.
Consider the effect of the displacement $\psi: x \rightsquigarrow \psi_\tau(x) = x+\tau$ on the graph of the function $y=f(x)$. The tranformed function is $\psi_\tau^* f$. Its value at any particular $x \in M = R^1$ is

$$\psi_\tau^* f(x) = f(\psi_\tau^{-1}(x)) = f(x-\tau)$$

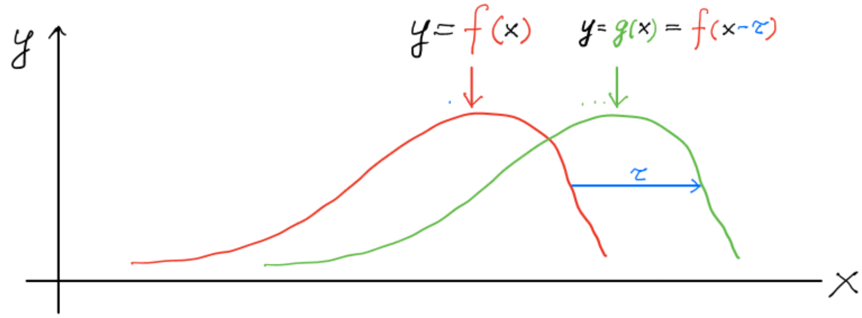

Figure 26.3 The point transformation (translation) $x \rightsquigarrow \psi_\tau(x) = x+\tau$ induces the functional transformation $f(x) \rightsquigarrow \psi_\tau^* f(x) = f(x-\tau)$. Here x must be some, but may be any element in $M = R^1$.

The τ-parametrized family of functions $\psi_\tau^* f$ is a curve in $C^\infty(M, R)$. The rate at which $\psi_\tau^* f$ changes is

$$\lim_{\tau \to 0} \frac{\psi_\tau^* f - f}{\tau} = \lim_{\tau \to 0} \frac{f \circ \psi_\tau^{-1} - f}{\tau} \equiv \mathcal{L}_u f$$

This is another scalar field on M. It is the <u>Lie derivative</u>

C26,7

of the scalar field f along the flow generated by the vector field $u = u^i \frac{\partial}{\partial x^i}$.

For the one-dimensional example in Figure 26.3 the Lie derivative of f along the tranlation generating $u = \frac{\partial}{\partial x}$ is

$$\mathcal{L}_u f = \lim_{\tau \to 0} \frac{f(\psi_\tau^{-1}(x)) - f(x)}{\tau}$$
$$= \lim_{\tau \to 0} \frac{f(\psi_{-\tau}(x)) - f(x)}{\tau}$$
$$= \lim_{\tau \to 0} \frac{f(x-\tau) - f(x)}{\tau} = -\frac{df}{dx}$$

Warning: On the internet there are many prominent entries where $\mathcal{L}_u f = \frac{df}{dx}$. This is because these entries are incorrect. Figure 26.3 shows where the minus sign comes from.

Problem 26.1 (Lie Transport of a Vector)

Given a scalar field $g(w)$ and a preexisting vector field $V = v^i \frac{\partial}{\partial w^i}$. It is a derivation. When evaluated on g at a typical point $w = \psi_t(w_0)$ depicted in Figure 26.4, the result is

$$v^i(w) \frac{\partial g(w)}{\partial w^i} = v^i(\psi_t(w_0)) \left.\frac{\partial g(w)}{\partial w^i}\right|_{w=\psi_t(w_0)}.$$

Referring to Figure 26.4, the problem is: compare the preexisting vector $v^i(w_0) \frac{\partial}{\partial w^i}$ with (the image of) the vector obtained by moving $v^i(w) \frac{\partial}{\partial x^i}$ from w by means of ψ_t^{-1} along the curve $c(w_0; t)$ back to its starting point w_0. This to-be-obtained image of $v^i(w) \frac{\partial}{\partial w^i}$ is a derivation at w_0. It is a linear combination of the basis vectors $\left.\frac{\partial}{\partial w^i}\right|_{w=w_0} \equiv \frac{\partial}{\partial w_0}$ at $w = w_0$. It is obtained using (i) the chain rule for diffentiating the composite of two functions and (ii) the fact

C26.8

that the Jacobian matrix of the inverse of a given one-to-one mapping equals the inverse of the Jacobian of that mapping:

$$\left[\frac{\partial\left(\psi_t^{-1}(w_0)\right)^i}{\partial w^j}\right] = \left[\frac{\partial\left(\psi_t(w)\right)^i}{\partial w_0^j}\right]^{-1}.$$

The action of $\mathbf{v}$ on $g \in C^\infty(M, w, R)$ results in the real number

$$\begin{aligned}
\mathbf{v}[g]_w &= v^i(w)\frac{\partial}{\partial w^i}\left[g(w)\right] \\
&= v^i(w)\,\frac{\partial\, g\left(\psi_t\circ\psi_t^{-1}(w)\right)}{\partial w^i} \qquad w_0 = \psi_t^{-1}(w), \quad w = \psi_t(w_0) \qquad (C26.2) \\
&= \underbrace{v^i\left(\psi_t(w_0)\right)\frac{\partial\left(\psi_t^{-1}(w)\right)^j}{\partial w^i}\bigg|_{w=\psi_t(w_0)}}\cdot\frac{\partial\, g\left(\psi_t(w_0)\right)}{\partial w_0^j} \qquad f = g\circ\psi_t \in C^\infty(M, w_0, R) \\
&= \bar{v}^j(w_0)\,\frac{\partial f(w_0)}{\partial w_0} \qquad (C26.3)
\end{aligned}$$

The vector $\bar{v}^j(w_0)\frac{\partial}{\partial w_0} \in T_{w_0}(M)$ at point w_0 is the Lie-dragged image of the vector $v^i(w)\frac{\partial}{\partial x^i} \in T_w(M)$ at point w.

Both (i) that Lie-dragged image and (ii) the preexisting vector $v^i(w_0)\frac{\partial}{\partial w_0^i}$ are elements of the same tangent space at w_0. In the limit as $t\to 0$ their difference gives rise to the Lie derivative $\mathcal{L}_u\mathbf{v}$ of $\mathbf{v}$. Indeed, from Eq.(26.2) and expanding ψ_t and ψ_{-t} to 1st order, one has

$$\begin{aligned}
\mathcal{L}_u\mathbf{v} &= \lim_{t\to 0}\frac{\bar{v}^j(w_0) - v^j(w_0)}{t}\frac{\partial}{\partial w_0^j} = \lim_{t\to 0}\frac{\left[v^i\left(\psi_t(w_0)\right)\frac{\partial\left(\psi_{-t}^j(w)\right)}{\partial w^i}\Big| - v^j(w_0)\right]\frac{\partial}{\partial w_0^j}}{t} \\
&= \lim_{t\to 0}\frac{\left[v^i(w_0^k + tu^k + \cdots)\frac{\partial(w^j - tu^j + \cdots)}{\partial w^i}\Big|_{w_0} - v^j(w_0)\right]\frac{\partial}{\partial w_0^j}}{t} \\
&= \lim_{t\to 0}\frac{1}{t}\left[\left(v^i(w_0) + \frac{\partial v^i}{\partial w^k}tu^k + \cdots\right)\left(\delta_i^j - t\frac{\partial u^j}{\partial w^i}\Big|_{w_0} + \cdots\right) - v^j(w_0)\right]\frac{\partial}{\partial w_0^j}
\end{aligned}$$

$$= \left[u^k(w) \frac{\partial v^j}{\partial w^k} - v^i(w) \frac{\partial u^j}{\partial w^i} \right]_{w^0} \frac{\partial}{\partial w^j} = [u, v] \qquad \text{(C26.9)}$$

The Lie derivative of the vector field v into the direction of a flow field generated by the vector field u is the commutator of of the two vector fields,

$$\boxed{\mathcal{L}_u v = [u, v]}$$

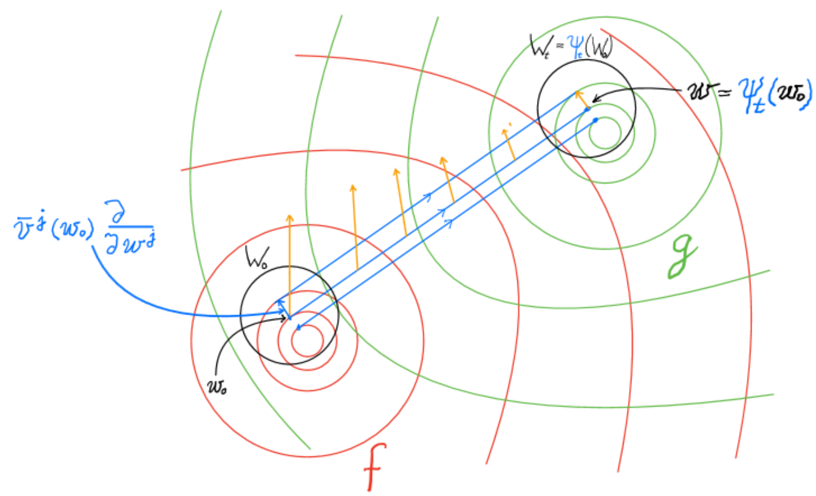

Figure 26.4 The preexisting vector $v^i(w) \frac{\partial}{\partial w^i}$ at the point $w = \Psi_t(w_0)$ got "Lie transported" by Ψ_t^{-1} along the curve $c(w_0, t)$ from w to $w_0 = \Psi_t^{-1}(w)$. The resulting vector is $\bar{v}^j(w_0) \frac{\partial}{\partial w_0}$, Eq(26.2).