LECTURE 26

I. Isograms of a function
vs
Curves of the flow field of a vector field

II. Taylor series on a curve of the flow field

III. The exponential map

IV. Vector as a displacement generator

In MTW read Box 8.4, Sect 9.2, 9.6

In Singer & Thorpe read p126 (or P142 in the Springer edition)

In mathematics the importance of a concept can be gauged by the number of contexts where it plays a central role. A vector u at P, because of its multi-faced nature is, such a concept. It is

(i) an element in $T_P(M)$ Lecture 23
(ii) a derivation at P Lecture 24
(iii) the tangent to a curve at P Lecture 25
(iv) a vectorial displacement generator Lecture 26
(v) the exponent in the exponential map $e^{\tau u}$ Lecture 26

I. Scalar Field vs. Vector Field

Salar properties on a manifold are mathematized by scalar functions on a manifold. If $f(x^1, \ldots, x^n)$ is a scalar function on M, then its physical referents are in the form of mathematical isograms.

By contrast, the flow field of a given vector field on M,

$$u(x) = u^i(x^1, \ldots, x^n)\frac{\partial}{\partial x^i},$$

is mathematized by means of the continuos family of orbits or trajectories, i.e. "integral curves," whose tangents are vectors of that preexisting vector field.

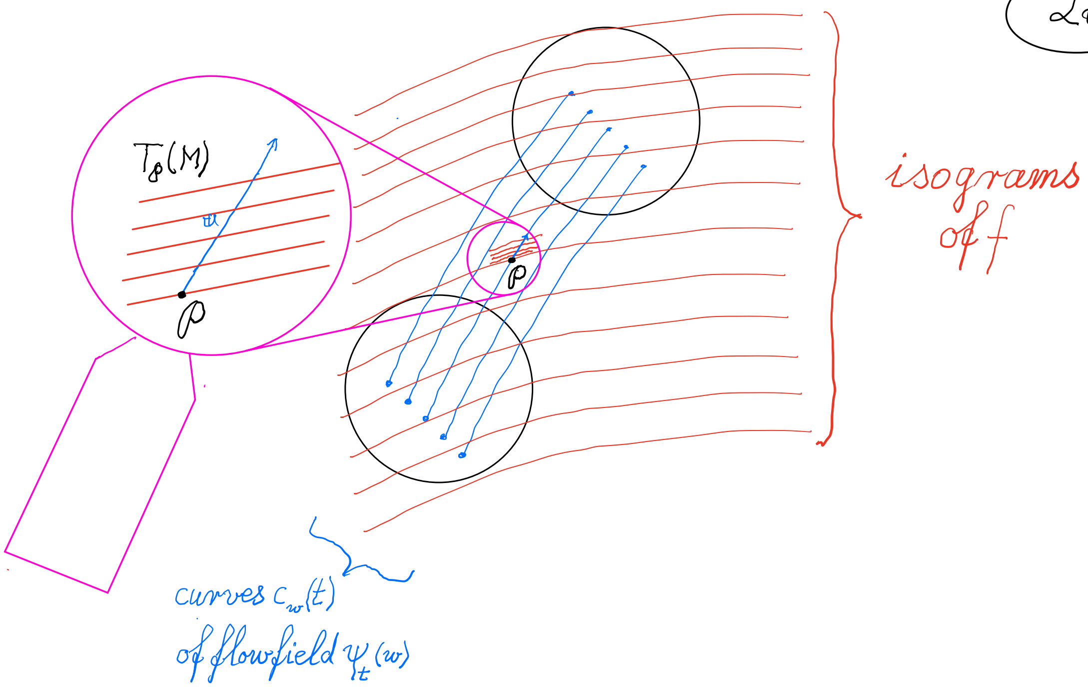


Figure 26.1: Integral curves $c_w(t)$ passing through the isograms of the scalar function $f(x)$.

The observed existence of scalar isograms and integral curves trace their origins back to the

Definition ("Scalar field")

A <u>smooth scalar field</u> $f \in C^\infty(M,R)$, is the assignment to each point $\mathcal{P} \in M$ a scalar $f(\mathcal{P}) \in R$; it is a family of $(n-1)$-dimensional <u>isograms</u> parametrized by a single variable, say $\gamma \in R$:

$\mathcal{P} \to \varphi(\mathcal{P})$

$$f(\mathcal{P}) = f_{rep}(\varphi(\mathcal{P})) \equiv f_{rep}(x^1(\mathcal{P}), \dots, x^n(\mathcal{P})) = \gamma .$$

$f_{rep}(x^1, \dots, x^n) = f \circ \varphi^{-1}(x^1, \dots, x^n)$

and is a smooth function the coordinatized points in M.

[Reminder from P 21.5: $f_{rep}(\overbrace{x^1,\dots,x^n}^{\varphi(\mathcal{P})}) = f(\overbrace{\varphi^{-1}(x^1,\dots,x^n)}^{\mathcal{P}}) = f_{rep}\circ\varphi\circ\varphi^{-1}(x^1,\dots,x^n)$]

Definition ("Flow field")

The <u>smooth flow field</u> of a smooth vector field $u \in C^\infty(M, T(M))$, namely the assignment to each point $\mathcal{P}\in M$ a vector $u(\mathcal{P}) \in T_{\mathcal{P}}(M)$, is a family of <u>integral curves</u> parametrized by $w \in R^n$:

$$c_w(t) = \{c^i(t; w^1, \dots, w^n\} = \{c^i_w(t)\}$$

II. Taylor series of a scalar function on a curve.

In non-linear mathematics a scalar field and a vector field are paired concepts corresponding to the duality between linear functions ("covectors") and vectors in linear mathematics.

This scalar-vector pairing is achieved by the method of the Taylor series as follows:

1.) Consider the solution curve, say $c(t)$, to the differential equation

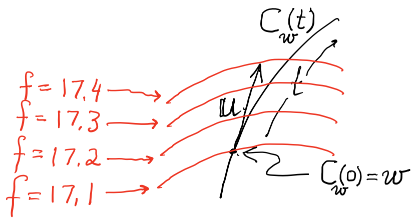
$$\frac{d}{dt} = u$$

i.e. to

$$\frac{dc^i(t)}{dt}\frac{\partial}{\partial x^i} = u^i(c^1(t), \dots, c^n(t))\frac{\partial}{\partial x^i}$$

with starting point

$$c_w(t{=}0) = w \equiv (w^1, \dots, w^n)$$



26.5

Figure 26.2: Integral curve c(t) passing through the isograms of f

2.) Consider a scalar field $f(x)$. Evaluate it on the solution curve $c_w(t)$ and obtain the single variable function

$$f(c_w(t)) = h(t)$$

3.) Consider its Taylor series expansion around $t=0$ at $c_w(0)=w$:

$$f(c_w(t)) = h(0) + t\left.\frac{dh}{dt}\right|_{t=0} + \frac{t^2}{2!}\frac{d^2h(0)}{dt^2} + \cdots$$

$$f(c_w^j(t)) = f(c_w^j(0)) + t\left.\frac{dc_w^i}{dt}\right|_{t=0}\left.\frac{\partial f}{\partial x^i}\right|_{c_w^i(0)} + \frac{t^2}{2}\left.\frac{d}{dt}\frac{d}{dt}h(t)\right|_{t=0}$$

In terms of the directional derivative

$$\frac{d}{dt} = \frac{dc^i}{dt}\frac{\partial}{\partial x^i} = u^i\frac{\partial}{\partial x^i} = D_{\mathbf{u}} = \mathbf{u}$$

one has

$$f(c_w^j(t)) = f(w) + t\,D_u f(w) + \frac{t^2}{2!}D_u D_u f(w) + \cdots$$

$$= f(w) + t\,\mathbf{u}f(w) + \frac{t^2}{2!}\mathbf{u}\mathbf{u}f(w) + \cdots$$

$$= \left(1 + t\mathbf{u} + \frac{t^2}{2}\mathbf{u}\mathbf{u} + \cdots\right)f(x)\Big|_{x=w}$$

Thus, the value f along any point along the curve $c_w(t)$ is

$$\boxed{f(c_w(t)) = \exp(t\mathbf{u})f(w)} \qquad (26.1)$$

(26.5b)

Comment 1.

Recall that $c_w(t)$ is a curve which starts at $w = c_w(0)$ and pushes this point forward to $c_w(t) = w'$, i.e. $\{c^i_w(t) = w'^i : i = 1, \ldots, n\}$. Let $f(w) = w^k$.

Apply Eq. 26.1 to this function. One finds that

$$\{w^k\}_{k=1}^n \rightsquigarrow \exp(tu)\{w^k\}_{k=1}^n = \{w'^k\}_{k=1}^n$$

or

$$\boxed{w \rightsquigarrow \exp(tu)\, w = w'} \qquad (26.1\,b)$$

Thus e^{tu} is the translation operator which pushes the initial point w $w = c_w(0)$ along the curve c_w to the point $w' = c_w(t)$.

Comment 2

The justification for introducing the exponential operator $e^{t\mathfrak{u}}$ comes from its additive property

$$\boxed{\exp\big((\tau+\tau')\mathfrak{u}\big) = \exp(\tau\mathfrak{u})\,\exp(\tau'\mathfrak{u})} \qquad (26.2)$$

From Eqs. (25.2) and (25.12) [in Lecture 25] we have

$$c_w(t) = \psi_t(w)$$

$$\psi_{\tau+\tau'}(w) = \psi_\tau \circ \psi_{\tau'}(w)$$

In order to verify Eq. (26.2), use Eq. (26.1) three times in conjunction with these two equations from Lecture 25.

$$\begin{aligned}
\exp\big((\tau+\tau')\mathfrak{u}\big) f(w) \overset{①}{=} f\big(c_w(\tau+\tau')\big) &= f\big(\psi_{\tau+\tau'}(w)\big) \\
&= f\circ\psi_{\tau+\tau'}(w) \\
&= f\circ\psi_\tau\circ\psi_{\tau'}(w) \\
&= f\circ\psi_\tau\big(\psi_{\tau'}(w)\big) \\
&= f\circ\psi_\tau\big(c_w(\tau')\big) \\
&\overset{②}{=} \exp(\tau'\mathfrak{u})\, f\circ\psi_\tau(w) \\
&= \exp(\tau'\mathfrak{u})\, f\big(\psi_\tau(w)\big) \\
&= \exp(\tau'\mathfrak{u})\, f\big(c_w(\tau)\big) \\
&\overset{③}{=} \exp(\tau'\mathfrak{u})\,\exp(\tau\mathfrak{u})\, f(w)
\end{aligned}$$

This holds for all smooth functions at the point $w \in M$, i.e. $\forall f \in C^\infty(M, w, \mathbb{R})$.
Thus Eq. (26.2) is true indeed.

IV. Vector as a displacement generator

Consider two points

$$C_w(0) = w = \mathcal{P}$$

$$C_w(\tau) = \overline{\mathcal{P}} \equiv \mathcal{P} + \Delta\mathcal{P} \qquad (\text{"displaced point"})$$

on the curve $C_w(t)$ whose tangent at $w = \mathcal{P}$ is

$$\mathbf{u} = u^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}\Big|_{\{x^j\}=\{w^j\}}$$

and the function

$$f(x^1, \dots, x^n) = x^k$$

with its two isograms $f = a^k$ and $f = \bar{a}^k = a^k + \Delta x^k$ running through these two points as depicted in Figure 26.3.

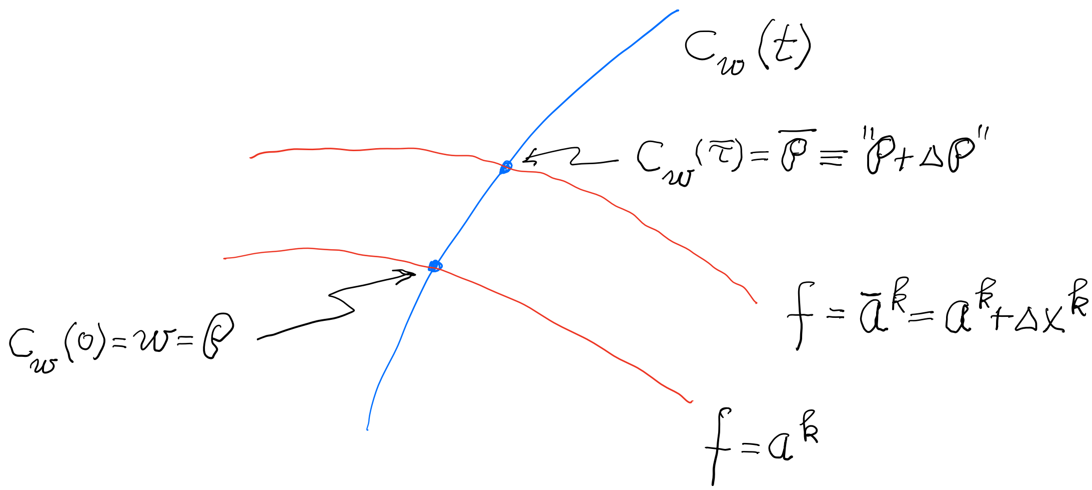


Figure 26.3: Two points, $\mathcal{P}$ and the displaced point $\overline{\mathcal{P}} = \mathcal{P} + \Delta\mathcal{P}$, on a given curve $C_w(t)$, with two isograms of the coordinate function $f(x^1, \dots, x^n) = x^k$ running through them.

The value of $f(\bar{P})$ is related to that of $f(P)$ by means of the Taylor series expansion

$$f(\bar{P}) = \exp(\bar{\tau} u) f(P)$$
$$f(P+\Delta P) = f(P) + \bar{\tau} u f(P) + \frac{(\bar{\tau})^2}{2!} u u f(P) + \cdots$$
$$= f(w^j) + \bar{\tau} u^i \frac{\partial f}{\partial x^i}\Big|_{\{w^\ell\}} + \frac{(\bar{\tau})^2}{2!} u^i \frac{\partial}{\partial x^i}\left(u^j \frac{\partial f}{\partial x^j}\right)\Big|_{\{w^\ell\}} + \cdots$$

Letting $f = x^k$, one obtains

$$x^k + \Delta x^k = x^k + \bar{\tau} u^i \frac{\partial x^k}{\partial x^i} + \frac{\bar{\tau}^2}{2!} u^i \frac{\partial}{\partial x^i}\left(u^j \frac{\partial x^k}{\partial x^j}\right) + \cdots$$

or

$$\Delta x^k = \bar{\tau} u^k + \frac{(\bar{\tau})^2}{2!} u^i \frac{\partial (u^k)}{\partial x^i} + \cdots$$

Thus $\{\Delta x^k\}$ are the coordinate differences of the <u>displacement</u> from P to $P+\Delta P$. If they lie on the integral curve whose tangent at P is u, then

$$\Delta x^k = \Delta P(x^k) = \bar{\tau}\, u(x^k) + \frac{(\bar{\tau})^2}{2!} u u (x^k) + \cdots,$$

or leaving the x^k's as-yet-unspecified,

$$\Delta P = \bar{\tau} u\big|_P + \underbrace{\frac{(\bar{\tau})^2}{2!} u u\big|_P}_{\text{neglegible for } |\bar{\tau}| \ll 1} + \cdots$$

The first term of the displacement ΔP is the <u>principal linear part</u>. It is a <u>vector</u>. For $|\bar{\tau}| \ll 1$ the 2^{nd} non-linear term is <u>not</u> a vector. This is because it is not a derivation:

$$u u(fg) \neq f\, u u(g) + u u(f)\, g.$$