

LECTURE 27

27.1

- I. Commutator of two vector fields
- II. Cotangent space of differentials
- III. Coordinate-induced dual basis
- IV. Physical meaning of a differential and of a tangent vector

In MTW read Box 9.2, Sections 9.6, 9.3, 9.4
Memorize Eq. (9.14).

I. Commutator $[u, v]$ of two vector fields 27.2

Consider two vector fields u and v and their intersecting integral curves $c_{u(\tau)}$ and $c_{v(\delta)}$ in a neighborhood of P .

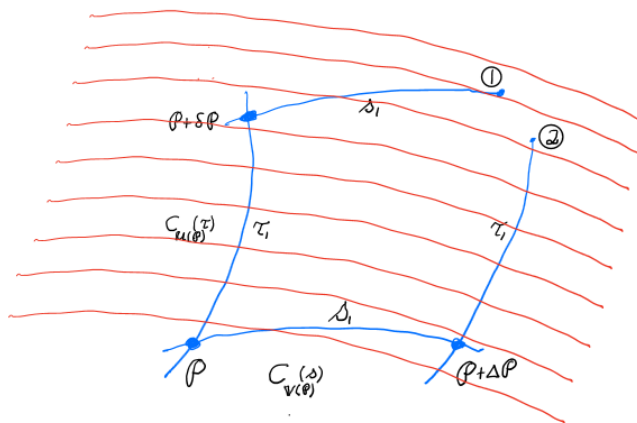


Figure 27.1: Integral curves $c_u(\tau)$ and $c_v(\delta)$ of vector fields u and v passing through the isograms of scalar function f .

Consider two of these curves which, as depicted in Figure 27.1, intersect at point P . Together with their respective opposing curve segments they form the quadrilateral $(P, P+\delta P, 1, 2, P+\Delta P)$. This quadrilateral is in general not closed. Its non-zero gap in Figure 27.1 is formed by the end points ① and ② of the two curve segments.

The environment of this open quadrilateral is permeated by a scalar function $f \in C^\infty(M, \mathbb{R})$ with typical isograms as depicted in Figure 27.1.

The system of two vector fields u and v together 27.3 with the set of ambient smooth scalar fields, such as f , gives rise to the concept of the commutator $[u, v]$ of u and v by asking and answering the following three questions:

1. $f(\textcircled{1}) = ?$
2. $f(\textcircled{2}) = ?$
3. $f(\textcircled{1}) - f(\textcircled{2}) = ?$

They are answered using the Taylor series expansion along

$$\text{and } \begin{array}{l} \textcircled{1} \rightarrow \mathcal{P} + \delta \mathcal{P} \rightarrow \mathcal{P} \\ \textcircled{2} \rightarrow \mathcal{P} + \Delta \mathcal{P} \rightarrow \mathcal{P} \end{array}$$

To 2nd order accuracy one has along $\textcircled{1} \rightarrow \mathcal{P} + \delta \mathcal{P} \rightarrow \mathcal{P}$

$$\begin{aligned} f(\textcircled{1}) &= f \Big|_{\mathcal{P} + \delta \mathcal{P}} + \delta, v f \Big|_{\mathcal{P} + \delta \mathcal{P}} + \frac{\delta^2}{2!} v v f \Big|_{\mathcal{P} + \delta \mathcal{P}} + \dots \\ &= f \Big|_{\mathcal{P}} + \tau, u f \Big|_{\mathcal{P}} + \frac{\tau^2}{2} u u f \Big|_{\mathcal{P}} + \dots \\ &\quad \delta, v f \Big|_{\mathcal{P}} + \delta, \tau, u v f \Big|_{\mathcal{P}} + \dots + \frac{\delta^2}{2} v v f \Big|_{\mathcal{P}} + \dots \end{aligned}$$

Similarly, to 2nd order one has along $\textcircled{2} \rightarrow \mathcal{P} + \Delta \mathcal{P} \rightarrow \mathcal{P}$

$$\begin{aligned} f(\textcircled{2}) &= f \Big|_{\mathcal{P} + \Delta \mathcal{P}} + \tau, u f \Big|_{\mathcal{P} + \Delta \mathcal{P}} + \frac{\tau^2}{2!} u u f \Big|_{\mathcal{P} + \Delta \mathcal{P}} + \dots \\ &= f \Big|_{\mathcal{P}} + \delta, v f \Big|_{\mathcal{P}} + \frac{\delta^2}{2} v v f \Big|_{\mathcal{P}} + \dots \\ &\quad \tau, u f \Big|_{\mathcal{P}} + \tau, \delta, v u f \Big|_{\mathcal{P}} + \dots + \frac{\tau^2}{2} u u f \Big|_{\mathcal{P}} + \dots \end{aligned}$$

$f(①) - f(②)$ defines a vector, the commutator $[u, v]$ at P :

$$\frac{f(①) - f(②)}{\Delta \tau} = (uv - vu) f \Big|_P \equiv [u, v] f \Big|_P$$

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Comment

(i) $[u, v]$ is a derivation and hence a vector:

$[u, v](fg) = f[u, v]g + [u, v](f)g$, which is the (to-be-verified) product ("Leibnitz") rule.

(ii) If $u = u^\alpha \frac{\partial}{\partial x^\alpha}$ and $v = v^\beta \frac{\partial}{\partial x^\beta}$, then $[u, v] = [u, v]^\delta \frac{\partial}{\partial x^\delta}$, where the component $[u, v]^\delta$ depend on the (to-be-computed) partial derivatives of u^α and v^β .

II. The Cotangent space $T_P^*(M)$

A) Vector Space

Every vector space V has its space of duals, the covector space V^* . Furthermore, given a basis $B = \{e_1, \dots, e_n\}$ for V , its dual basis $B^* = \{\omega^1, \dots, \omega^n\}$ is determined by the duality relation,

$$\langle \omega^j | e_i \rangle = \delta^j_i.$$

Every point P of a manifold has its tangent vector space

$$V_P = T_P(M).$$

Relative to a coordinate system $x^\delta(P) = r^\delta \cdot \varphi(P)$ the induced basis for $T_P(M)$ is

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$$B_p = \left\{ \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\}$$

so that

$$T_p(M) = \text{span} \left\{ \frac{\partial}{\partial x^i} \right\}$$

B) Space of Duals to $T_p(M)$

The space $T_p^*(M)$ of duals to $T_p(M)$ is the space of differentials at p . A differential arises from observations which are condensed into the following

Definition ("The differential of a scalar function")

Consider $u, v \in T_p(M)$

$$f, g \in C^\infty(M, \mathbb{R})$$

$$\alpha, \beta \in \mathbb{R};$$

then the differential df of f is the following linear function

$$df: T_p(M) \longrightarrow \mathbb{R}$$

$$u \rightsquigarrow \langle df | u \rangle = u(f)$$

Note 1: df is linear on $T_p(M)$. Indeed,

$$\langle df | \alpha u + \beta v \rangle = (\alpha u + \beta v)(f)$$

$$= \alpha u(f) + \beta v(f)$$

$$= \alpha \langle df | u \rangle + \beta \langle df | v \rangle$$

$\therefore df \in T_p^*(M)$, the set of linear functions on $T_p(M)$

Note 2: The operator d [which maps functions to linear functions on $T_p(M)$] is linear, i.e. the differential of a linear combination of

functions equals a linear combination of their differentials. 27.6
 This observation is mathematizes by the following

Theorem ("Linearity of d ")

$$d(\alpha f + \beta g) = \alpha df + \beta dg$$

proof: $\langle d(\alpha f + \beta g) | u \rangle = u(\alpha f + \beta g)$
 $= \alpha u(f) + \beta u(g)$
 $= \alpha \langle df | u \rangle + \beta \langle dg | u \rangle \quad \forall u \in T_p(M)$
 $= \langle \alpha df + \beta dg | u \rangle \quad \because \text{lin. functions are}$
closed under lin.
combinations

This holds $\forall u \in T_p(M)$. Thus,

$$\alpha df + \beta dg = d(\alpha f + \beta g)$$

Consequently, a linear combination of differentials is another differential in $T_p^*(M)$. Thus $T_p^*(M)$, the space of differentials, is a vector space, the cotangent space.

III. Dual basis for $T_p^*(M)$

$$\text{Let } f^1(x^1, \dots, x^n) = x^1$$

$$\vdots$$

$$f^n(x^1, \dots, x^n) = x^n$$

Conclusion: $\{dx^1, \dots, dx^n\}$ is a basis for $T_p^*(M)$.

a) $\{dx^j\}$ has the spanning property

(27.7)

Indeed, let $df \in T_p^*(M)$. Then

$$\begin{aligned} \langle df | u \rangle &= u(f) = u^i \frac{\partial f}{\partial x^i} \\ &= u^i \delta_i^j \frac{\partial f}{\partial x^j} \\ &= u^i \frac{\partial x^j}{\partial x^i} \frac{\partial f}{\partial x^j} \\ &= \frac{\partial f}{\partial x^i} \langle dx^j | \frac{\partial}{\partial x^i} \rangle u^i \\ &= \langle \frac{\partial f}{\partial x^i} dx^j | u \rangle \quad \forall u \in T_p(M) \end{aligned}$$

Thus, $df = \frac{\partial f}{\partial x^i} dx^i$, i.e. $\{dx^j\}$ is a spanning set

b) The set $\{dx^j\}$ is linearly independent

Consider

$$\alpha_j dx^j = 0.$$

Evaluate it on $e_i = \frac{\partial}{\partial x^i}$ and obtain

$$\begin{aligned} 0 &= \langle 0 | e_i \rangle = \langle \alpha_j dx^j | \frac{\partial}{\partial x^i} \rangle \\ &= \alpha_j \delta_i^j \\ &= \alpha_i \quad i = 1, \dots, n \end{aligned}$$

Conclusion

$B^* = \{dx^1, \dots, dx^n\}$ is the coordinate induced basis for $T_p^*(M)$

which is dual to the basis

$B = \{e_1 = \frac{\partial}{\partial x^1}, \dots, e_n = \frac{\partial}{\partial x^n}\}$ for $T_p(M)$ and

$$\langle dx^j | e_i \rangle = \langle dx^j | \frac{\partial}{\partial x^i} \rangle = \delta_i^j$$

which is the duality relationship.

IV. Physical meaning of df and u .

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Evaluate f on the curve $c(t) = \{c^i(t)\}$ whose tangent at $p = c(t)$ is $\dot{c}(t) = \left. \frac{dc^i}{dt} \frac{\partial}{\partial x^i} \right|_p = u^i \left. \frac{\partial}{\partial x^i} \right|_p = u$, and find that

$$\frac{d(f)}{dt} = \underbrace{\frac{\partial(f)}{\partial x^i}}_{(1)} u^i = \langle df | u \rangle = u(f). \quad (2)$$

① Omitting explicit reference to u , one has

$df =$ "rate of change of f into an as-yet-unspecified direction".

② Omitting explicit reference to f , one has

$u = \frac{d}{dt} =$ "instantaneous rate of change (at a point on a curve) of an as-yet-unspecified scalar function".