Lecture 28: Algebraic Supplement

28.0

1. Lagrange's expansion by complementary minors
2. Associativity of the exterior product

In MTW read Exercise 4.12 ("Symmetry Operations as Tensors")

(S28.1)

Some Key Algebraic Methods for Exterior Algebra

The exterior product and associativity

$$(\sigma^p \wedge \sigma^q) \wedge \sigma^r = \sigma^p \wedge (\sigma^q \wedge \sigma^r) \equiv \sigma^p \wedge \sigma^q \wedge \sigma^r$$

is a manifestation of Lagrange's expansions of a determinant by complementary minors. The fundamental building blocks for these expansions are the determinants

$$\begin{vmatrix} \delta^{\alpha_1}_{\mu_1} & \delta^{\alpha_2}_{\mu_1} & \cdots & \delta^{\alpha_p}_{\mu_1} \\ \delta^{\alpha_1}_{\mu_2} & \delta^{\alpha_2}_{\mu_2} & \cdots & \delta^{\alpha_p}_{\mu_2} \\ \vdots & & & \\ \delta^{\alpha_1}_{\mu_p} & \delta^{\alpha_2}_{\mu_p} & \cdots & \delta^{\alpha_p}_{\mu_p} \end{vmatrix} \equiv \delta^{\alpha_1 \cdots \alpha_p}_{\mu_1 \cdots \mu_p} . \qquad (28.1)$$

They comprise the generalized Kroneker permutation symbol. Its values are (obviously!)

$$\delta^{\alpha_1 \cdots \alpha_p}_{\mu_1 \cdots \mu_p} = \begin{cases} 1 & \text{whenever } \alpha_1, \cdots, \alpha_p \text{ is an } \underline{\text{even}} \text{ permutation of the integers } \mu_1, \cdots, \mu_p. \\ -1 & \text{whenever } \alpha_1 \cdots \alpha_p \text{ is an } \underline{\text{odd}} \text{ permutation of the integers } \mu_1, \cdots, \mu_p \\ 0 & \text{if any two integers in } \alpha_1, \cdots, \alpha_p \text{ repeat} \\ 0 & \text{if } \alpha_1, \cdots, \alpha_p \text{ and } \mu_1, \cdots, \mu_p \text{ are different sets of integers,} \end{cases} \qquad (28.2)$$

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It satisfies the following identities

$$\delta^{\alpha_1\cdots\alpha_p}_{\mu_1\cdots\mu_p} = \delta^{\alpha_1\;\cdots\;\alpha_p}_{\gamma_1\cdots\gamma_q\gamma_{q+1}\cdots\gamma_p}\underbrace{\delta^{|\gamma_1\cdots\gamma_q|}_{\mu_1\cdots\mu_q}\;\delta^{|\gamma_{q+1}\cdots\gamma_p|}_{\mu_{q+1}\cdots\mu_p}}_{\text{complementary minors}}$$

As an example, consider the two-by-two determinant

$$\begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & \delta^{\alpha_2}_{\beta_1} \\ \delta^{\alpha_1}_{\beta_2} & \delta^{\alpha_2}_{\beta_2} \end{vmatrix} = \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1} = \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2} - \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}$$

$$= \delta^{\alpha_1\alpha_2}_{\beta_1\beta_2} = \begin{cases} +1 & \beta_1\beta_2 \text{ is an even permutation of } \alpha_1\alpha_2 \\ -1 & \text{" " " odd " " "} \\ 0 & \text{for any repeats} \\ 0 & \{\alpha_1,\alpha_2\} \neq \{\beta_1,\beta_2\} \end{cases}$$

$$1 \leq \alpha_i, \beta_i \leq n$$

The expansion of a 3×3 by minors reads as follows:

$$\begin{vmatrix} \delta^{\alpha_1}_{\beta_1} & \delta^{\alpha_2}_{\beta_1} & \delta^{\alpha_3}_{\beta_1} \\ \delta^{\alpha_1}_{\beta_2} & \delta^{\alpha_2}_{\beta_2} & \delta^{\alpha_3}_{\beta_2} \\ \delta^{\alpha_1}_{\beta_3} & \delta^{\alpha_2}_{\beta_3} & \delta^{\alpha_3}_{\beta_3} \end{vmatrix} \equiv \delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = \begin{cases} +1 & \text{even} \\ -1 & \text{odd} \\ 0 & \text{repeats} \\ 0 & \text{different sets} \end{cases}$$

$$= \sum_{\gamma_1<\gamma_2}\sum_{\gamma_3} \delta^{\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3}\,\delta^{|\gamma_1\gamma_2|}_{\beta_1\beta_2}\,\delta^{\gamma_3}_{\beta_3}$$

$$= \sum_{\gamma_1}\sum_{\gamma_2<\gamma_3} \delta^{\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3}\,\delta^{\gamma_1}_{\beta_1}\,\delta^{|\gamma_2\gamma_3|}_{\beta_2\beta_3}$$

Using this expansion, one concludes that the wedge product

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$$
\begin{aligned}
\delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\otimes\sigma^{\beta_3} &= \delta^{\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3}\left(\delta^{|\gamma_1\gamma_2|}_{\beta_1\beta_2}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\right)\otimes\delta^{\gamma_3}_{\beta_3}\,\sigma^{\beta_3}\\
&\equiv \left(\sigma^{\alpha_1}\wedge\sigma^{\alpha_2}\right)\wedge\sigma^{\alpha_3}\\
&= \delta^{\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3}\,\delta^{\gamma_1}_{\beta_1}\,\sigma^{\beta_1}\otimes\left(\delta^{|\gamma_2\gamma_3|}_{\beta_2\beta_3}\,\sigma^{\beta_2}\otimes\sigma^{\beta_3}\right)\\
&= \sigma^{\alpha_1}\wedge\left(\sigma^{\alpha_2}\wedge\sigma^{\alpha_3}\right)
\end{aligned}
$$

is associative. (Here the α_i's are free indeces; the others are summation indeces.)

Example 1:

Question: How does one construct the exterior product of, say, a 1-form and a 2-form?

Answer: (i) Let $\sigma^{\gamma_1}\wedge\sigma^{\gamma_2} = \sigma^{\gamma_1}\otimes\sigma^{\gamma_2} - \sigma^{\gamma_2}\otimes\sigma^{\gamma_1} = \underbrace{\delta^{\gamma_1\gamma_2}_{\beta_1\beta_2}}\sigma^{\beta_1}\otimes\sigma^{\beta_2}$

$$
\begin{vmatrix}\delta^{\gamma_1}_{\beta_1} & \delta^{\gamma_2}_{\beta_1}\\ \delta^{\gamma_1}_{\beta_2} & \delta^{\gamma_2}_{\beta_2}\end{vmatrix} = \begin{cases} +1 & \text{the } \gamma\text{'s are an \underline{even} perm'n of the } \beta\text{'s}\\ -1 & \text{" " \underline{odd}}\\ 0 & \text{set of } \gamma\text{'s} \neq \text{set of } \beta\text{'s}\\ 0 & \text{there is a repeat in the } \gamma\text{'s or the } \beta\text{'s}\end{cases}
$$

be a 2-form.

(ii) Let $\sigma^{\alpha_3} = \delta^{\alpha_3}_{\beta_3}\sigma^{\beta_3}$

Then $\sigma^{\alpha_1}\wedge\sigma^{\alpha_2}\wedge\sigma^{\alpha_3}$ is defined as follows:

(a) $$\left(\sigma^{\alpha_1}\wedge\sigma^{\alpha_2}\right)\wedge\sigma^{\alpha_3} \equiv \delta^{\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3}\Big[\underbrace{\left(\delta^{|\gamma_1\gamma_2|}_{\beta_1\beta_2}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\right)}_{\boxed{\sigma^{\gamma_1}\wedge\sigma^{\gamma_2}}}\otimes\,\delta^{\gamma_3}_{\beta_3}\,\sigma^{\beta_3}\Big]$$

$$= \delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\otimes\sigma^{\beta_3}$$

(b) $$\sigma^{\alpha_1}\wedge\left(\sigma^{\alpha_2}\wedge\sigma^{\alpha_3}\right) \equiv \delta^{\alpha_1\alpha_2\alpha_3}_{\gamma_1\gamma_2\gamma_3}\Big[\delta^{\gamma_1}_{\beta_1}\,\sigma^{\beta_1}\otimes\left(\delta^{|\gamma_2\gamma_3|}_{\beta_2\beta_3}\,\sigma^{\beta_2}\otimes\sigma^{\beta_3}\right)\Big]$$

$$= \delta^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\otimes\sigma^{\beta_3}$$

(c) (a) together with (b) show that $\wedge$ is associative:

$$\left(\sigma^{\alpha_1}\wedge\sigma^{\alpha_2}\right)\wedge\sigma^{\alpha_3} = \sigma^{\alpha_1}\wedge\left(\sigma^{\alpha_2}\wedge\sigma^{\alpha_3}\right) \equiv \sigma^{\alpha_1}\wedge\sigma^{\alpha_2}\wedge\sigma^{\alpha_3}$$

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Example 2

Let $\sigma^{\alpha_1}\wedge\sigma^{\alpha_2} = \sigma^{\alpha_1}\otimes\sigma^{\alpha_2} - \sigma^{\alpha_2}\otimes\sigma^{\alpha_1} \equiv \delta^{\alpha_1\alpha_2}_{\beta_1\beta_2}\,\omega^{\beta_1}\otimes\omega^{\beta_2}$

Let $\sigma^{\alpha_3}\wedge\sigma^{\alpha_4}\wedge\sigma^{\alpha_5} = \Sigma$ even permutations of $\omega^{\alpha_3}\otimes\omega^{\alpha_4}\otimes\omega^{\alpha_5}$

$-\Sigma$ odd " " "

$$\equiv \delta^{\alpha_3\alpha_4\alpha_5}_{\beta_3\beta_4\beta_5}\,\sigma^{\beta_3}\otimes\sigma^{\beta_4}\otimes\sigma^{\beta_5}$$

Then

$$(\sigma^{\alpha_1}\wedge\sigma^{\alpha_2})\wedge(\sigma^{\alpha_3}\wedge\sigma^{\alpha_4}\wedge\sigma^{\alpha_5}) \equiv$$

$$\equiv \delta^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}_{\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5}\left[\delta^{|\gamma_1\gamma_2|}_{\beta_1\beta_2}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\otimes\left(\delta^{|\gamma_3\gamma_4\gamma_5|}_{\beta_3\beta_4\beta_5}\,\omega^{\beta_3}\otimes\omega^{\beta_4}\otimes\omega^{\beta_5}\right)\right]$$

$$= \delta^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}_{\beta_1\beta_2\beta_3\beta_4\beta_5}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\otimes\sigma^{\beta_3}\otimes\sigma^{\beta_4}\otimes\sigma^{\beta_5}$$

From this definition one sees that $\wedge$ is associative.
Thus one can write the wedge product in the unambiguous and unique way as

$$\sigma^{\alpha_1}\wedge\sigma^{\alpha_2}\wedge\sigma^{\alpha_3}\wedge\sigma^{\alpha_4}\wedge\sigma^{\alpha_5} = \delta^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}_{\beta_1\beta_2\beta_3\beta_4\beta_5}\,\sigma^{\beta_1}\otimes\sigma^{\beta_2}\otimes\sigma^{\beta_3}\otimes\sigma^{\beta_4}\otimes\sigma^{\beta_5}$$

$$\left(= \delta^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}_{\beta_1\beta_2\beta_3\beta_4\beta_5}\,\sigma^{\beta_1}\wedge\sigma^{\beta_2}\wedge\sigma^{\beta_3}\wedge\sigma^{\beta_4}\wedge\sigma^{\beta_5}/5!\right)$$

$$\left(= \delta^{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}_{|\beta_1\beta_2\beta_3\beta_4\beta_5|}\,\sigma^{\beta_1}\wedge\sigma^{\beta_2}\wedge\sigma^{\beta_3}\wedge\sigma^{\beta_4}\wedge\sigma^{\beta_5}\right)$$

CONCLUSION:

The wedge product is an expression of the expansion of a 5×5 determinant in terms of its 2×2 and 3×3 complementary minors.

Some comments on Problem 4.12 in MTW and on the "wedge" product.

The tensor "Alt" is a (multi) linear map.

Let $V = V_{\mu_1 \cdots \mu_p}\, \omega^{\mu_1} \otimes \cdots \otimes \omega^{\mu_p}$ be a tensor of rank $\binom{0}{p}$. S28.5

Let
$$V_{[\mu_1 \cdots \mu_p]} = \frac{1}{p!} \sum_{\pi} (-1)^{\pi} V_{\pi(\mu_1, \cdots, \mu_p)} = \frac{1}{p!} \delta^{\alpha_1 \cdots \alpha_p}_{\mu_1 \cdots \mu_p} V_{\alpha_1 \cdots \alpha_p}$$

where $\delta^{\alpha_1 \cdots \alpha_p}_{\mu_1 \cdots \mu_p}$ is the generalized Kronecker permutation symbol, whose origin and definition were given by Eqs. (28.1) and (28.2) on page 28.1.

Example:
$$V_{[\mu_1 \mu_2 \mu_3]} = \frac{1}{6}\Big(V_{\mu_1\mu_2\mu_3} + V_{\mu_2\mu_3\mu_1} + V_{\mu_3\mu_1\mu_2} - V_{\mu_3\mu_2\mu_1} - V_{\mu_2\mu_1\mu_3} - V_{\mu_1\mu_3\mu_2}\Big)$$

The definition of "Alt" is as follows:

$$\mathrm{Alt}: \ \left\{\binom{0}{p}\right\} \equiv \begin{pmatrix}\text{space of tensors}\\ \text{of rank}\binom{0}{p}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{space of antisymmetric}\\ \text{tensors}\end{pmatrix} \subset \left\{\binom{0}{p}\right\}$$

$$V \rightsquigarrow \mathrm{Alt}\, V = V_{[\mu_1 \cdots \mu_p]}\, \omega^{\mu_1} \otimes \cdots \otimes \omega^{\mu_p}$$

where
$$\{V_{\mu_1 \cdots \mu_p}\} \rightsquigarrow \left\{(\mathrm{Alt}\, V)_{\mu_1 \cdots \mu_p}\right\} = \left\{V_{[\mu_1 \cdots \mu_p]}\right\}$$

It follows that Alt is the identity map when its domain is restricted to the subspace Λ^p of p-forms. Indeed, let let $V_{\mu_1 \cdots \mu_p} = V_{[\mu_1 \cdots \mu_p]}$. Then

$$(\mathrm{Alt}\, V)_{\mu_1 \cdots \mu_p} = \frac{1}{p!} \delta^{\alpha_1 \cdots \alpha_p}_{\mu_1 \cdots \mu_p} V_{[\alpha_1 \cdots \alpha_p]}$$
$$= \frac{1}{p!} \delta^{\alpha_1 \cdots \alpha_p}_{\mu_1 \cdots \mu_p}\, \frac{1}{p!} \delta^{\beta_1 \cdots \beta_p}_{\alpha_1 \cdots \alpha_p} V_{\beta_1 \cdots \beta_p}$$

$$= \sum_{\alpha_1<\cdots<\alpha_p}\cdots\sum \frac{1}{p!}\,\delta^{\alpha_1\cdots\alpha_p}_{\mu_1\cdots\mu_p}\,\frac{p!}{p!}\,\delta^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p}\,V_{\beta_1\cdots\beta_p}$$

$$= \frac{1}{p!}\,\delta^{\beta_1\cdots\beta_p}_{\mu_1\cdots\mu_p}\,V_{\beta_1\cdots\beta_p}$$

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$$= V_{[\mu_1\cdots\mu_p]} \underset{\text{"given"}}{=} V_{\mu_1\cdots\mu_p}$$

Thus Alt is the identity map on antisymmetric tensors indeed.