LECTURE 29

29.1

I. Parallel transport between tangent spaces.

II. Mathematization of parallel transport

III Mathematization relative to a coordinate basis.

In MTW read Sect. 8.3, 8.5; 10.3, 10.4; Box 10.2, 10.3

NOTICE: For typographical efficiency Lecture 29 denotes the tangent space $T_{\mathcal{P}}(M)$ at a point $\mathcal{P}$ by $M_{\mathcal{P}}$. Thus

$$T_{\mathcal{P}}(M) \equiv M_{\mathcal{P}}$$

29.2

I. Parallel transport.

Each point P of a manifold has a vector space, denoted by $T_P(M)$ or simply by M_P. This is the set of vectors tangent to their respective curves through P. Each of vector spaces is called a tangent space $(T_P(M))$ of the manifold at point P.

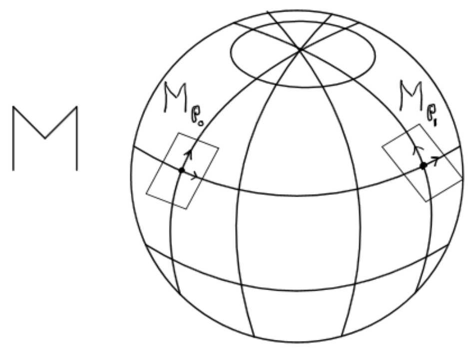

Figure 29.1: Each of two points P_0 and P_1 in the manifold M have its tangent space $M_{P_0}(=T(M_{P_0}))$ and $M_{P_1}(=T(M_{P_1}))$.

Although each point of a manifold has its own vector space of tangent vectors, there is no "natural"* isomorphism between different vector spaces.

*\footnote{"natural" = uniquely defined, non-arbitrary.}

The concept parallelism, and hence the concept of parallel transport is a geometrical structure which does provide a natural isomorphism. It mathematizes and generalizes what is observed in the real world. A parallel transport is also called a connection.

Example 1 ("Schild's Ladder")

Given a curve, smooth or broken, parallel transport is introduced via straight lines and congruent triangles with matching sides

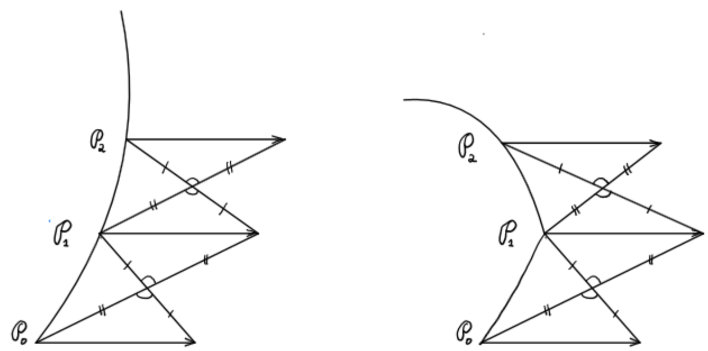

Figure 29.2: Parallel transport of a vector along a given smooth curve (a) and a broken one (b) via Schild's Ladder construction.

Example 2 ("Inherited Parallelism")

Given: Manifold M is a subspace of an ambient flat space with its pre-exting law of parallel transport.

M inherits this law. This inheritance, which is based on the ambient parallelism, is a two-step process:

(i) parallel translate all elements of tangent space M_{P_0} at P_0 to a nearby point P_1 on M.

(ii) Project these parallel translates onto M at P_1.

This two-step mapping is an isomorphism from M_{P_0} into and onto M_{P_1}. The principal linear part of this mapping depends linearly on the separation between the two points.

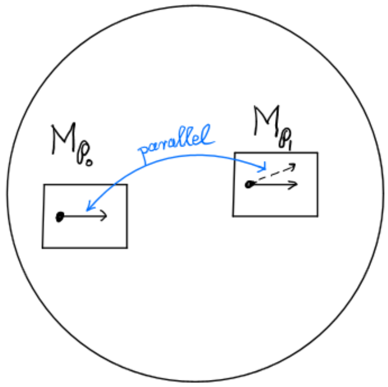

Figure 29.3: Parallelism between vectors in two nearby vector spaces on the two-sphere S^2 imbedded in E^3 as depicted in Figure 29.1. The parallel image of M_{P_0} at P_1 gets projected onto M_{P_1}. This projection is the parallel translate of M_{P_0} into its nearby neighbor M_{P_1}. For small displacements $P_0 \to P_1$ the length of a vector does not change under the projection.

II. Mathematization of Parallel Transport

Parallel transport of vectors in M_P to vectors in $M_{P+\Delta P}$ is expressed by an isomorphism between M_P and $M_{P+\Delta P}$.

Let P and $P+\Delta P$ be connected by an infinitesimal t-parametrized curve segment whose tangent is the vector u.

Let $\{e_i\}$ and $\{\bar{e}_i\}$ be bases for M_P and $M_{P+\Delta P}$ respectively.

The displacement vector ΔP connecting P and $P+\Delta P$ by curve parameter Δt is

$$\Delta P = u \Delta t = e_i u^i \Delta t \equiv e_i \Delta x^i. \qquad (29.1)$$

Thus

$$\Delta x^i = x^i(P+\Delta P) - x^i(P^i)$$

is the coordinate difference between the points $\mathcal{P}$ and $\mathcal{P}+\Delta\mathcal{P}$, which are connected by the curve segment $[0, \Delta t]$.

The parallel-transport induced isomorphism between $M_{\mathcal{P}}$ and $M_{\mathcal{P}+\Delta\mathcal{P}}$, which is depicted in Figure 29.4, is as follows:

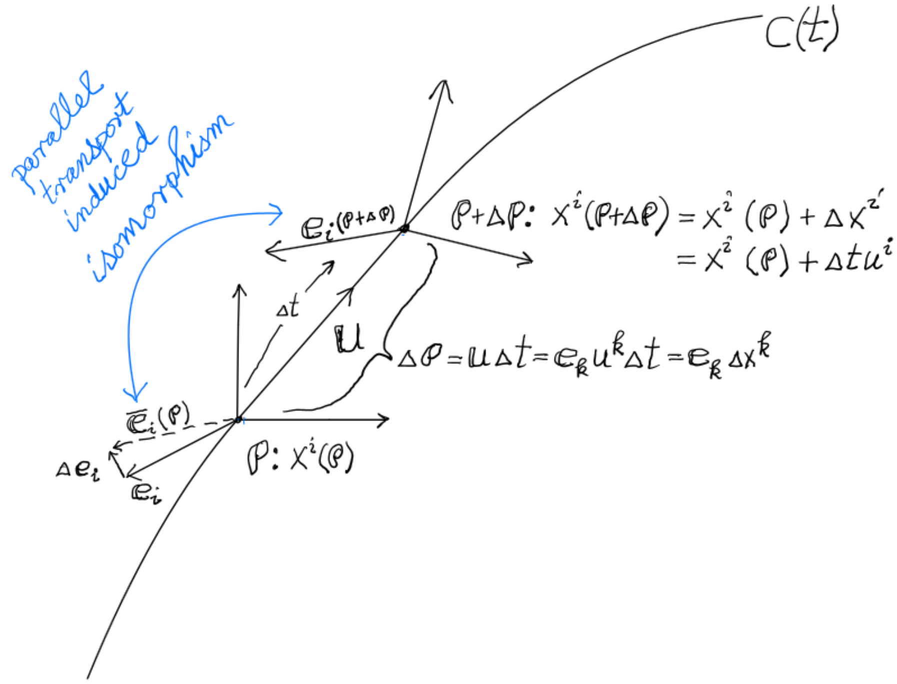

Figure 29.4: Two instantaneous frames, $\{\mathbf{e}_i(\mathcal{P})\}$ at point $\mathcal{P}$ and $\{\mathbf{e}_i(\mathcal{P}+\Delta\mathcal{P})\}$ at $\mathcal{P}+\Delta\mathcal{P}$, are related by means of the parallel-transport-induced isomorphism.

Each of the three vectors in the equation

$$\Delta \mathbf{e}_i(\mathcal{P}) = \bar{\mathbf{e}}_i(\mathcal{P}) - \mathbf{e}_i(\mathcal{P}),$$

(i) $\bar{\mathbf{e}}_i(\mathcal{P})$, the parallel translate of $\mathbf{e}_i(\mathcal{P}+\Delta\mathcal{P})$,

(ii) $\mathbb{e}_i(\mathcal{P})$, the pre-existing basis vector, and 29.6

(iii) $\Delta\mathbb{e}_i$, their difference

belongs to the same vector space $T_{\mathcal{P}}(M)$ at the point $\mathcal{P}$.

The parallel-transport-mapping is an isomorphism (i.e. a linear one-to-one transformation) between adjacent tangent spaces:

$$M_{\mathcal{P}+\Delta\mathcal{P}} \longrightarrow M_{\mathcal{P}}$$

$$\mathbb{e}_i(\mathcal{P}+\Delta\mathcal{P}) \rightsquigarrow \mathbb{e}_i + \Delta\mathbb{e}_i = \mathbb{e}_j\left(\delta^j{}_i + \omega^j{}_i(\Delta)\right)$$

$$\underset{\substack{\uparrow \\ i^{th}\text{entry}}}{(0,\cdots,0,1,0,\cdots,0)} \rightsquigarrow \Big(\omega^1{}_i(\Delta),\cdots,\omega^{i-1}{}_i(\Delta),\,1+\underset{\substack{\uparrow \\ \text{no sum}}}{\omega^i{}_i(\Delta)},\,\omega^{i+1}{}_i(\Delta),\cdots,\omega^n{}_i(\Delta)\Big)$$

When the two tangent spaces coincide, i.e. $\Delta\mathcal{P}=0$, the isomorphism is the identity and $\omega^j{}_i(\Delta)=0$ $(i,j=1,\cdots,n)$

(29.7)

For typographical shorthand we are writing

$$\boxed{\omega^j{}_i\,(\Delta\mathcal{P}) \equiv \omega^j{}_i(\Delta)}\,.$$

The matrix representation of this isomorphism is

$$\left[\delta^i{}_j + \omega^i{}_j(\Delta)\right] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} \omega^1{}_1(\Delta) & \omega^1{}_2(\Delta) & \cdots & \omega^1{}_n(\Delta) \\ \vdots & \vdots & & \vdots \\ \omega^n{}_1(\Delta) & \omega^n{}_2(\Delta) & \cdots & \omega^n{}_n(\Delta) \end{bmatrix}$$

The matrix $[\omega^i{}_j(\Delta)]$ generates the isomorphism and it depends linearly on $\Delta\mathcal{P}$, the separation between $\mathcal{P}$ and $\mathcal{P}+\Delta\mathcal{P}$, Eq. (29.1) on page (29.4).

These two attributes are condensed into the statements that

(1) $\Delta e_i = e_j\,\omega^j{}_i(\Delta)$ is the vectorial amount by which the basis vector $e_i \in M_{\mathcal{P}}$ deviates from being parallel to $e_i(\mathcal{P}+\Delta\mathcal{P}) \in M_{\mathcal{P}+\Delta\mathcal{P}}$, and that

(2) this vectorial deviation depends linearly on the displacement vector $\Delta\mathcal{P} = u\,\Delta t$:

$$\Delta e_i \equiv e_j\,\omega^j{}_i\,(u\,\Delta t) = e_j\,\omega^j{}_i\,(u)\,\Delta t$$

Mathematize this linear dependence by introducing the array of covectors $\underset{\sim}{\omega}^j{}_i$, the array of connection one-forms, characterized by the requirement that they yield the expansion coefficients

$$\langle \underset{\sim}{\omega}^j{}_i \,|\, u\,\Delta t\rangle = \omega^j{}_i(u\,\Delta t)\ \left(=\omega^j{}_i(u)\,\Delta t\right)\ \forall\, u \in M_{\mathcal{P}}$$

and hence yield

$$\Delta e_i = e_j\,\langle \underset{\sim}{\omega}^j{}_i \,|\, u\,\Delta t\rangle. \qquad (29.2)$$

By leaving the vector u as-yet-unspecified, the parallel transport relation between $M_{\mathcal{P}+\Delta\mathcal{P}}$ and $M_{\mathcal{P}}$ is mathematized by

$$\boxed{\underset{\sim}{d}e_i = e_j\,\underset{\sim}{\omega}^j{}_i}$$

or more explicitly (29.8)

$$\boxed{\underset{\sim}{d}\mathbf{e}_i = \mathbf{e}_j \otimes \omega^j{}_i}.$$

This is a vectorial one-form equation. Infer its geometrical meaning by evaluating both sides on the tangent vector $\mathbf{u}$ and referring to Eq. (29.2):

$$\langle \underset{\sim}{d}\mathbf{e}_i | \mathbf{u} \rangle = \mathbf{e}_j \langle \underset{\sim}{\omega}^j{}_i | \mathbf{u} \rangle = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{e}_i}{\Delta t} \equiv \frac{d\mathbf{e}_i}{dt}$$

This is the rate of change of $\mathbf{e}_i$ away from parallelism due to motion into the direction of $\mathbf{u}$. Equivalently,

$d\mathbf{e}_i$ = rate of change (relative to parallel transport) of $\mathbf{e}_i$ into an as-yet-unspecified direction
= parallel transport-induced vectorial 1-form.
$= \mathbf{e}_j \otimes \underset{\sim}{\omega}^j{}_i$

III. Parallel transport mathematized relative to a coordinate induced basis.

The law of parallel transport, $\underset{\sim}{d}\mathbf{e}_i = \mathbf{e}_j \otimes \underset{\sim}{\omega}^j{}_i$, is quite general. Let us apply it to the particular case of a coordinate induced basis and its dual,

$$\left\{\mathbf{e}_i = \frac{\partial}{\partial x^i}\right\}; \quad \left\{\underset{\sim}{\omega}^k = dx^k\right\}.$$

Expand each of the connection 1-forms $\underset{\sim}{\omega}^j{}_i$ in terms of the dual basis elements:

$$\underset{\sim}{\omega}^j{}_i = \Gamma^j{}_{ik}\, dx^k = \Gamma^j{}_{i1}\, dx^1 + \cdots + \Gamma^j{}_{in}\, dx^n$$

The (coordinate dependent) coefficients $\Gamma^j{}_{ik}(x^\ell)$ are the "Christoffel symbols of the second kind".

Expand the tangent vector $\mathbf{u}$, the direction of motion, in terms

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of the coordinate basis,

$$\mathbf{u} = u^\ell \frac{\partial}{\partial x^\ell},$$

and find

$$\langle d\mathbf{e}_i | \mathbf{u} \rangle = \langle \mathbf{e}_j\, \omega^j{}_i | \mathbf{u} \rangle$$

$$= \langle \mathbf{e}_j \Gamma^j{}_{ik}\, \underset{\sim}{d}x^k \,|\, u^\ell \frac{\partial}{\partial x^\ell} \rangle$$

$$= \mathbf{e}_j \Gamma^j{}_{ik}\, u^\ell \underbrace{\frac{\partial x^k}{\partial x^\ell}}_{\delta^k_\ell} = \mathbf{e}_j \Gamma^j{}_{ik}\, u^\ell \delta^k_\ell$$

$$= \mathbf{e}_j \Gamma^j{}_{ik}\, u^k$$

Thus $\lim\limits_{\Delta t \to 0} \frac{\Delta \mathbf{e}_i}{\Delta t} = \langle d\mathbf{e}_i | \mathbf{u} \rangle = \mathbf{e}_j \Gamma^j{}_{ik} u^k$

= rate of change (relative to parallel xport) of $\mathbf{e}_i$ due to motion into the direction of $\mathbf{u}$.

The indeces of $\Gamma^j{}_{ik}$ have a specific significance:

k - direction of motion

i - "which basis vector" is deviating from parallelism

j - expansion index, a summation ("dummy") index.

Next: I. The covariant differential of a vector $\mathbf{v} = v^\ell \mathbf{e}_\ell$

II. The covariant derivative " " " "