30.1

LECTURE 30

I. The Covariant Differential

In MTW read Section 14.5

II. The Covariant Derivative

In MTW read Sections 8.3, 8.5, 10.3, 10.4
Box 10.2, 10.3

I. The Covariant Differential

30.2

The parallel-transport-given isomorphism between adjacent tangent spaces is condensed into the vectorial differentials

$$d e_i = e_j \otimes \underset{\sim}{\omega}^j{}_i \qquad (30.1)$$

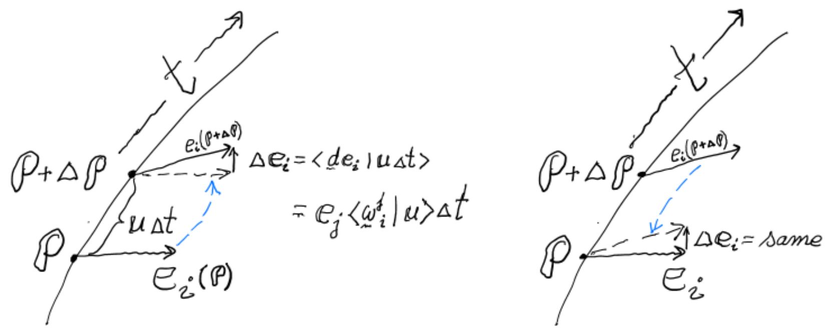

Figure 30.1: Along a t-parametrized curve a preexisting t-parametrized family of basis vectors deviates from being parallel.

Relative to the coordinate cotangent basis $\{dx^i\}$ the connection one-forms

$$\underset{\sim}{\omega}^j{}_i = \Gamma^j{}_{ik}\, dx^k$$

are given in terms of the functions

$$\Gamma^i{}_{jk}(P) = \Gamma^i_{jk}(x^1, \dots, x^n),$$

the Christoffel symbols of the 2nd kind. The law of parallel transport of vectors and tensors is mathematized by these functions.

30.3

Given a smooth vector field

$$\mathbb{v} = v^i(x^1, \dots, x^n)\,\mathbb{e}_i ,$$

how does one extend this law to such a vector field?
The means of parallel transporting a vector is to apply the product rule governing a differential. A differential is based on taking differences, in this case between a pre-existing vector at a point and the parallel translate from any one of its neighbors as depicted in Figure 30.2,

$$\Delta \mathbb{V} = \mathbb{V}_{P+\Delta P} - \left(\text{parallel transport of } \mathbb{V} \text{ from } T_P(M) \text{ to } T_{P+\Delta P}(M)\right)$$
$$= \mathbb{V}_P \quad - \left(\text{parallel transport of } \mathbb{V} \text{ from } T_{P+\Delta P}(M) \text{ to } T_P(M)\right)$$

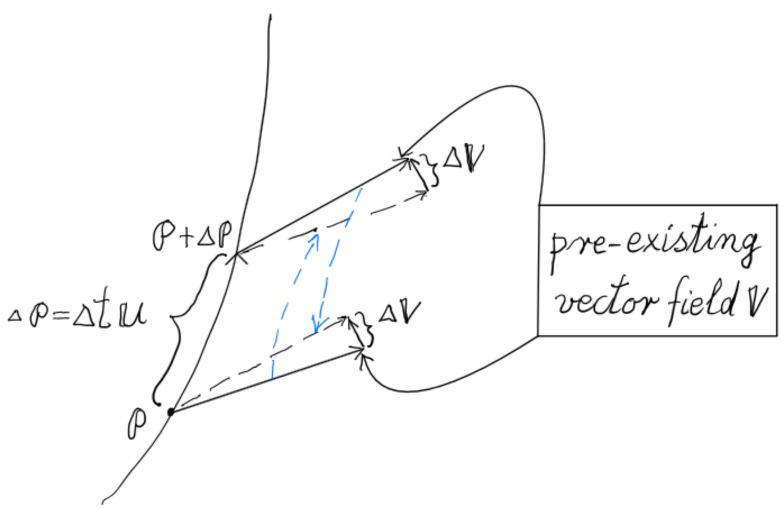

Figure 30.2: A pre-existing non-parallel vector field $\mathbb{v}$ deviates (by $\Delta \mathbb{V}$) from parallelism (stiped blue arrows) among vectors at near-by points P and $P+\Delta P$.

(30.4)

This diffence is well-defind because it is of two vectors at the same point. In addition, the dominant part of its difference depends only linearly on the displacement vector $\Delta \mathcal{P} = \Delta t\, \mathbf{u}$ separating $\mathcal{P}+\Delta\mathcal{P}$ from $\mathcal{P}$. Ignoring higher order dependencies and keeping the principal linear part of that difference, one has at point $\mathcal{P}$

$$\mathbf{v}(\mathcal{P}+\Delta\mathcal{P}) - \begin{pmatrix}\text{parallel} \\ \text{translate} \\ \text{of } \mathbf{v} \text{ from } \mathcal{P}\end{pmatrix} = \Delta \mathbf{v}_{\mathcal{P}+\Delta\mathcal{P}} = \Delta \mathbf{v}_{\mathcal{P}} = \Delta(\mathbf{e}_i v^i) = \mathbf{e}_i \Delta v^i + \Delta \mathbf{e}_i\, v^i \qquad (30.2)$$

The 1st term is the change in $\mathbf{v}$ due the change in the scalar expansion coefficient v^i of $\mathbf{v}$, the 2nd is due to the non-parallelism of the basis vectors $\mathbf{e}_i$. Recall that

(i) the deviation of $\mathbf{e}_i$ from parallelism is

$$\Delta \mathbf{e}_i = \langle \mathbf{e}_j\, \underset{\sim}{\omega}^j{}_i \,|\, \mathbf{u} \rangle t$$
$$\equiv \langle d\mathbf{e}_i | \mathbf{u} \rangle t$$

and that

(ii) the Taylor series of v^i, namely, $v^i(\underbrace{c(\Delta t)}_{\mathcal{P}+\Delta t u_{\mathcal{P}}}) = v^i(\underbrace{c(0)}_{\mathcal{P}}) + \Delta t\, \mathbf{u}(v^i)\Big|_{\mathcal{P}=c(0)} + \frac{(\Delta t)^2}{2!} \mathbf{u}\mathbf{u}\, v^i\Big|_{\mathcal{P}=c(0)} + \cdots$

gives us

$$\Delta v^i \equiv v^i(\mathcal{P} + \Delta t\, \mathbf{u}_{\mathcal{P}}) - v^i(\mathcal{P})$$
$$= \Delta t\, \mathbf{u}_{\mathcal{P}}(v^i) + \frac{\Delta t^2}{2} \mathbf{u}_{\mathcal{P}} \mathbf{u}_{\mathcal{P}}(v^i) + \cdots \quad \text{(Lecture 26, page 26.8)}$$
$$= \Delta t\, \langle d v^i | \mathbf{u} \rangle + \quad '' \quad \text{(Lecture 25, page 25.5)}$$

Applying these facts to the deviation $\Delta\mathbf{v}$, Eq. (30.2) on page 30.4, one finds that the dominant linear contribution to $\Delta\mathbf{v}$ is

$$\Delta \mathbf{v} = \mathbf{e}_j \langle \underset{\sim}{d} v^j | \mathbf{u} \rangle \Delta t + \langle \underset{\sim}{d} \mathbf{e}_i | \mathbf{u} \rangle v^i \Delta t$$

Condense these observations and calculations (namely (i) and (ii)) into the following

30,5

Definition ("Covariant differential of a vector")

In moving from $\mathcal{P}$ to $\mathcal{P}+\Delta\mathcal{P}$, let $\Delta\mathbb{v}$ as given by Eq. (30.1) on page 30.2 be the principal linear part of the deviation of the vector field $\mathbb{v}=\mathbb{e}_i v^i$ from parallelism.

Then the vector-valued linear map

$$d\mathbb{v}: T_{\mathcal{P}}(M) \longrightarrow T_{\mathcal{P}}(M)$$

$$u\Delta t \rightsquigarrow \langle d\mathbb{v}|u\Delta t\rangle = \langle \mathbb{e}_i\, \underset{\sim}{d}v^i + d\mathbb{e}_i v^i | u\Delta t\rangle \equiv \Delta\mathbb{v}$$

is called the <u>covariant differential of $\mathbb{v}$.</u> This differential is a vector valued 1-form which, in light of Eq. (30.1) on page 30.2, is

$$\boxed{d\mathbb{v} = \mathbb{e}_i \otimes \left(\underset{\sim}{d}v^i + \underset{\sim}{\omega}{}^i{}_j\, v^j\right) \equiv \mathbb{e}_i\left(\underset{\sim}{d}v^i + \underset{\sim}{\omega}{}^i{}_j\, v^j\right).} \qquad (30.3)$$

It is a tensor of rank $\binom{1}{1}$.

Comment 1.

This definition is general; it applies regardless of one's choice of basis. Relative to a coordinate-indued basis and its dual,

$$\{\mathbb{e}_i = \tfrac{\partial}{\partial x^i}\} \quad \text{and} \quad \{dx^k\},$$

the covariant differential of $\mathbb{v}=v^i\mathbb{e}_i$ is

$$d\mathbb{v} = \mathbb{e}_i\left(\frac{\partial v^i}{\partial x^k}dx^k + \Gamma^i{}_{jk}\,dx^k\, v^j\right)$$

$$\boxed{d\mathbb{v} = \mathbb{e}_i \underbrace{\left(\frac{\partial v^i}{\partial x^k} + \Gamma^i{}_{jk}\, v^j\right)}_{v^i{}_{;k}} dx^k} \qquad (30.4)$$

The components $v^i{}_{;k}$ are those of a tensor of rank $\binom{1}{1}$, and

(30.6)

they are called the <u>components</u> of the <u>covariant derivative</u> of v.

Comment 2

The derivative

$$\lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \langle \underset{\sim}{d}\mathbf{v}|\mathbf{u}\rangle \qquad (30.4)$$

is v's rate of deviation from parallelism during motion into the direction u.

This process is concretized by an airplane flying along a great circle from New York to London. The interior of the airplane has its three basis vectors bolted to the aircraft.

They point forward-backward, up-down, and left-right. The aircraft carries a compass and its needle always points to the North. As the aircraft flies with non-zero velocity along its path the compass needle will rotate with non-zero angular velocity relative to the aircraft's basis vectors. This is because these vectors deviate from being parallel to the North Star direction. They rotate relative to the direction of the compass needle. The law governing this rotation is given by Eq. (30.4).

Comment 3.

By not specifying, but nevertheless implying, the direction

30.7

of motion in Eq.(30.4), namely $\frac{d\mathbb{V}}{dt} = \langle d\mathbb{V} | \mathbb{u} \rangle$, one is led to the concept of $\underset{\sim}{d}\mathbb{V}$ as "the rate of deviation from parallelism due to motion into an as-yet-unspecified direction."

Comment 4.

In one's motion along the direction $\mathbb{u}$, the vector $\mathbb{v} = \mathbb{e}_j v^j$ suffers from two causes in its rate of change as expressed by

$$\langle \underset{\sim}{d}\mathbb{v} | \mathbb{u} \rangle = \mathbb{e}_j \left(\frac{\partial v^j}{\partial x^k} + \Gamma^j{}_{ik} v^i \right) u^k$$

(i) The components v^j change, which is mathematized by the directional derivative,

$$D_{\mathbb{u}} v^j = \frac{\partial v^j}{\partial x^k} u^k$$

(ii) The basis vectors get altered (rotated, expanded, etc). This results in an additional change, one due to the change of the basis away from parallelism. It is in the form of the transformation

$$\left[\langle \underset{\sim}{\omega}^j{}_i | \mathbb{u} \rangle \right] v^i = \left[\Gamma^j{}_{ik} u^k \right] v^i. \qquad j = 1, \cdots, n$$

II. The Covariant Derivative

The covariant differential $\underset{\sim}{d}\mathbb{V}$, Eqs.(30.3)-(30.4), of the vector field $\mathbb{v}$ is the portal to extending the concept of the directional derivative of a scalar field to that of a vector field.

30.8

A) Let f be a scalar field. Recall (Lecture 23) that the differential df

(i) is defined by $\langle df | \mathbf{u} \rangle = \mathbf{u}(f)$

(ii) is linear: $\langle df | \mathbf{u}_1 + \mathbf{u}_2 \rangle = \langle df | \mathbf{u}_1 \rangle + \langle df | \mathbf{u}_2 \rangle$

$$\langle df | g\mathbf{u} \rangle = g \langle df | \mathbf{u} \rangle \qquad g \in C^\infty(M, \mathbb{R})$$

(iii) satisfies the product rule: $\langle d(fg) | \mathbf{u} \rangle = \langle df | \mathbf{u} \rangle g + f \langle dg | \mathbf{u} \rangle \qquad g \in C^\infty(M, \mathbb{R})$

B) Let $\mathbf{v} = \mathbf{e}_i v^i$ be a vector field. Recall from Page 30.5 that the differential $d\mathbf{v}$

(i) is defined by $\langle d\mathbf{v} | \mathbf{u} \rangle = \langle \mathbf{e}_j \otimes \underset{\sim}{d} v^j + \mathbf{e}_j \otimes \omega^j{}_i v^i | \mathbf{u} \rangle$

(ii) is linear : $\langle \underset{\sim}{d}\mathbf{v} | \mathbf{u}_1 + \mathbf{u}_2 \rangle = \langle d\mathbf{v} | \mathbf{u}_1 \rangle + \langle d\mathbf{v} | \mathbf{u}_2 \rangle$

$$\langle d\mathbf{v} | g\mathbf{u} \rangle = g \langle d\mathbf{v} | \mathbf{u} \rangle \qquad g \in C^\infty(M, \mathbb{R})$$

(iii) satisfies the product rule: $\langle d(f\mathbf{v}) | \mathbf{u} \rangle = f \langle d\mathbf{v} | \mathbf{u} \rangle + \langle df | \mathbf{u} \rangle \mathbf{v}$

C) Both df and $d\mathbf{v}$ share the linearity and the product rule property when these scalar/vectorial one-forms are evaluated on any $\mathbf{u} \in T_p(M)$. This commonality implies the definition of a (directional) derivative operator operator whose domain includes both scalar and vector field. This is achieved by the following definition and thereby condenses the essentials of Parallel tranport theory into 4 statements

Definition ("$\nabla_{\mathbf{u}} \mathbf{v}$")

(1) $\nabla_{\mathbf{u}}(\mathbf{v}_1 + \mathbf{v}_2) = \nabla_{\mathbf{u}} \mathbf{v}_1 + \nabla_{\mathbf{u}} \mathbf{v}_2$ "distributivity"

(2) $\nabla_{\mathbf{u}_1 + \mathbf{u}_2} \mathbf{v} = \nabla_{\mathbf{u}_1} \mathbf{v} + \nabla_{\mathbf{u}_2} \mathbf{v}$ "distributivity"

30.9

(3) For any scalar field f

$$\nabla_{f\mathbf{u}} \mathbf{V} = f \nabla_{\mathbf{u}} \mathbf{V}$$

(4) $\nabla_{\mathbf{u}}(f\mathbf{V}) = \nabla_{\mathbf{u}}(f)\,\mathbf{V} + f \nabla_{\mathbf{u}} \mathbf{V}$

"pointwise linearity"

"product ("Leibnitz") rule"

These four equations are to be compared and contrasted with the corresponding directional derivatives $\nabla_{\mathbf{u}} g$ of the scalar field g:

(1) $\nabla_{\mathbf{u}}(g_1 + g_2) = \nabla_{\mathbf{u}} g_1 + \nabla_{\mathbf{u}} g_2$

(2) $\nabla_{\mathbf{u}_1 + \mathbf{u}_2} g = \nabla_{\mathbf{u}_1} g + \nabla_{\mathbf{u}_2} g$

(3) For any scalar field f

$$\nabla_{f\mathbf{u}} g = f \nabla_{\mathbf{u}} g$$

(4) $\nabla_{\mathbf{u}}(fg) = \nabla_{\mathbf{u}}(f)\,g + f \nabla_{\mathbf{u}} g$

Comment

1. The operator ∇ is called the <u>covariant derivative</u> operator.
2. The operator d is called the <u>covariant differential</u> operator.
3. "$\nabla = d$"

Cartan before WWII.

Post WWII.