LECTURE 31: Appendix

Parallel Transport in the Polar-coordinatized Euclidean Plane.

SKIP the rest of this even though it is OK,

IV. Parallel Transport in the Polar-coordinatized Euclidean Plane

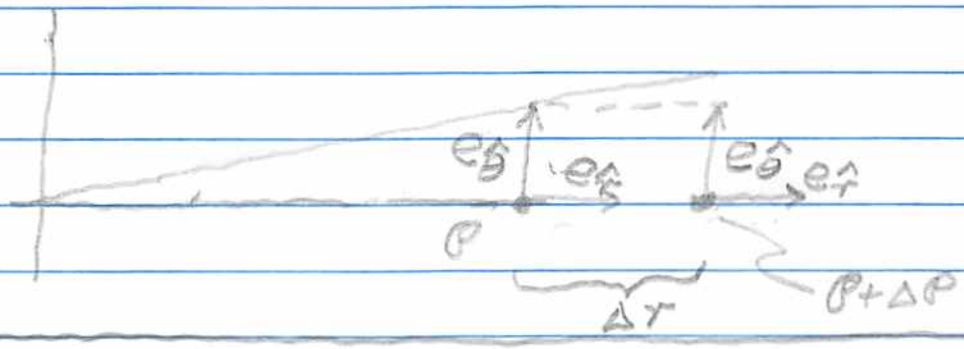
GIVEN:

(a) Coordinate basis $\{e_r = \frac{\partial}{\partial r}, e_\theta = \frac{\partial}{\partial \theta}\}$

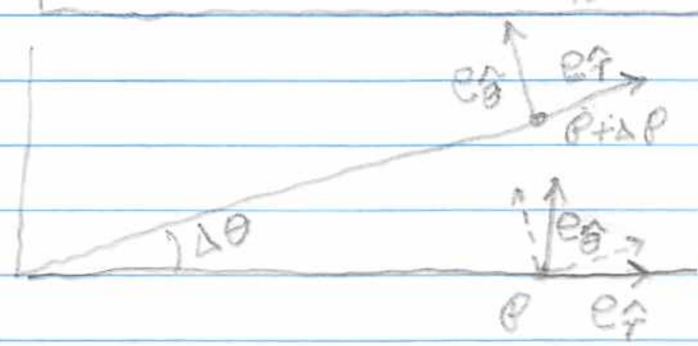
(b) Normalized basis $\{e_{\hat{r}} = \frac{\partial}{\partial r}, e_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}\}$

(c) Euclidean plane with its familiar parallel transport law

(i) $\boxed{\begin{aligned} \Delta e_{\hat{\theta}} &= 0 \\ \Delta e_{\hat{r}} &= 0 \end{aligned}}$ for $P \to P + \Delta P$: $(r, \theta) \to (r + \Delta r, \theta)$ $\begin{cases} (1) \\ (2) \end{cases}$



(ii) $\boxed{\begin{aligned} \Delta e_{\hat{\theta}} &= -e_{\hat{r}}\,\Delta\theta \\ \Delta e_{\hat{r}} &= e_{\hat{\theta}}\,\Delta\theta \end{aligned}}$ for $P \to P + \Delta P$: $(r, \theta) \to (r, \theta + \Delta\theta)$ $\begin{cases} (3) \\ (4) \end{cases}$



FIND:

$\nabla_{e_k} e_i = \Gamma^j{}_{ik} = ?$; $\underset{\sim}{d} e_i = e_j \otimes \underset{\sim}{\omega}^j{}_i$, $\nabla_{e_{\hat{k}}} e_{\hat{i}} = ?$
$= e_j \otimes (?)^j{}_i$

SOLUTION:

For each of the two translations apply the product rule to $e_\theta = r e_{\hat{\theta}}$ and to $e_r = e_{\hat{r}}$

(i) $\Delta P = \Delta \tau u = \Delta r \frac{\partial}{\partial r} + 0 \cdot \frac{\partial}{\partial \theta}$
$\Delta P : (\Delta r, 0)$

(1) First consider the change Δe_θ in e_θ as one moves into the e_r direction by an amount Δr.
On one hand one has

$$\Delta e_\theta = \Delta r \nabla_{e_r} e_\theta = \left(e_\theta \Gamma^\theta{}_{\theta r} + e_r \Gamma^r{}_{\theta r}\right) \Delta r$$

i.e.

$$\left.\frac{\Delta e_\theta}{\Delta r}\right)_\theta = \nabla_{e_r} e_\theta = e_\theta \Gamma^\theta{}_{\theta r} + e_r \Gamma^r{}_{\theta r} \qquad (5a)$$

On the other hand one has

$$\boxed{\begin{aligned}\Delta e_\theta &= \Delta(r e_{\hat{\theta}}) = \Delta r\, e_{\hat{\theta}} + r \underbrace{\Delta e_{\hat{\theta}}}_{=0} \\ &= \Delta r\, e_{\hat{\theta}} = \Delta r \frac{1}{r} e_\theta\end{aligned}}$$

GIVEN Ppty of E^2

so that

$$\left.\frac{\Delta e_\theta}{\Delta r}\right)_\theta = \frac{1}{r} e_\theta + 0 \cdot e_r \qquad (5b)$$

Comparing Eqs (5a) and (5b) one obtains

$$\nabla_{e_r} e_\theta = e_\theta \Gamma^\theta{}_{\theta r} + e_r \Gamma^r{}_{\theta r} = \frac{1}{r} e_\theta + 0 \cdot e_r$$

Thus one has

$$\boxed{\nabla_{e_r} e_\theta = e_\theta \frac{1}{r} + e_r \cdot 0}$$

Taking advantage of the linear independence of the basis vectors, one also obtains

$$\boxed{\Gamma^\theta{}_{\theta r} = \frac{1}{r} \quad \Gamma^r{}_{\theta r} = 0}$$

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(2) Secondly consider the change Δe_r in e_r

On one hand one has

$$\Delta e_r = (\Delta r) \nabla_{e_r} e_r = (e_\theta \Gamma^\theta{}_{rr} + e_r \Gamma^r{}_{rr}) \Delta r$$

$$\left.\frac{\Delta e_r}{\Delta r}\right)_\theta = \nabla_{e_r} e_r = e_\theta \Gamma^\theta{}_{rr} + e_r \Gamma^r{}_{rr}$$

On the other hand one has

$$\boxed{\Delta e_r = \underbrace{\Delta e_{\hat{r}}}_{=0} \leftarrow \boxed{\text{GIVEN}} \text{ ppty of } E^2}$$

$$\left.\frac{\Delta e_r}{\Delta r}\right)_\theta = 0$$

Consequently

$$\boxed{\nabla_{e_r} e_r = 0}$$

and

$$\boxed{\Gamma^\theta{}_{rr} = 0 \quad \Gamma^r{}_{rr} = 0}$$

(ii) $\Delta P = \Delta\tau\, \mathbb{u} = 0 \cdot \frac{\partial}{\partial r} + \Delta\theta \frac{\partial}{\partial \theta}$

$\Delta P : (0, \Delta\theta)$

(1) Again, first consider the change Δe_θ in e_θ.

On one hand one has

$$\Delta e_\theta = \Delta\theta\, \nabla_{e_\theta} e_\theta = \left(e_\theta \Gamma^\theta_{\theta\theta} + e_r \Gamma^r_{\theta\theta}\right)\Delta\theta$$

i.e $\left.\frac{\Delta e_\theta}{\Delta\theta}\right)_r = \nabla_{e_\theta} e_\theta = e_\theta \Gamma^\theta_{\theta\theta} + e_r \Gamma^r_{\theta\theta}$

On the other hand one has

$$\Delta e_\theta = \Delta(r e_{\hat\theta}) = \underbrace{\Delta r}_{\text{zero}}\, e_{\hat\theta} + r \underbrace{\Delta e_{\hat\theta}}_{-e_{\hat r}\Delta\theta = -e_r \Delta\theta}$$

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i.e

$$\left.\frac{\Delta e_\theta}{\Delta\theta}\right)_r = 0 \cdot e_\theta - r e_r$$

Consequently,

$$\boxed{\nabla_{e_\theta} e^\theta = -r e_r}$$

and

$$\boxed{\Gamma^\theta_{\theta\theta} = 0 \quad \Gamma^r_{\theta\theta} = -r}$$

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(2) Secondly, consider the change Δe_r in e_r

$$\Delta e_r = \Delta e_{\hat r} \quad = (e_\theta \Gamma^\theta_{r\theta} \cdot e_r \Gamma^r_{r\theta})\Delta\theta$$

as one moves into the e_θ direction by an amount $\Delta\theta$

Consequently,

$$\left.\frac{\Delta e_r}{\Delta \theta}\right)_r = e_\theta \Gamma^\theta{}_{r\theta} + e_r \Gamma^r{}_{r\theta}. \qquad (*)$$

Alternatively using the definition of $e_{\hat{r}}$ and its given behaviour and motion into the θ-direction (keeping r fixed), one has

$$\Delta e_r = \Delta e_{\hat{r}} = e_{\hat{\theta}} \Delta\theta \underset{\boxed{\text{GIVEN}}}{=} \frac{1}{r} e_\theta \Delta\theta,$$

so that

$$\left.\frac{\Delta e_r}{\Delta \theta}\right)_r = \frac{1}{r} e_\theta.$$

Comparison with Eq.(*) at the top of this page yields

$$\boxed{\Gamma^\theta{}_{r\theta} = \frac{1}{r} \qquad \Gamma^r{}_{r\theta} = 0}$$

Summary:

Relative to the polar coordinate basis the Christoffel symbols are

$$\boxed{\begin{array}{llll}
\nabla_{e_r} e_\theta : e_\theta, \Delta r & : \Gamma^\theta{}_{\theta r} = \frac{1}{r} & ; \Gamma^r{}_{\theta r} = 0 \\
\nabla_{e_r} e_r : e_r, \Delta r & : \Gamma^\theta{}_{rr} = 0 & ; \Gamma^r{}_{rr} = 0 \\
\nabla_{e_\theta} e_\theta : e_\theta, \Delta\theta & : \Gamma^\theta{}_{\theta\theta} = 0 & ; \Gamma^r{}_{\theta\theta} = -r \\
\nabla_{e_\theta} e_r : e_r, \Delta\theta & : \Gamma^\theta{}_{r\theta} = \frac{1}{r} & ; \Gamma^r{}_{r\theta} = 0
\end{array}}$$

These Christoffel not only define the directional derivatives of the basis vector but also their differentials. In light of $\underset{\sim}{\omega}^j{}_i = \Gamma^j{}_{ik} \underset{\sim}{d}x^k$, one has

$$\underset{\sim}{\omega}^\theta{}_\theta = \frac{1}{r}\underset{\sim}{d}r \quad ; \underset{\sim}{\omega}^\theta{}_r = \frac{1}{r}\underset{\sim}{d}\theta$$

$$\underset{\sim}{\omega}^r{}_\theta = -r\underset{\sim}{d}\theta \quad ; \underset{\sim}{\omega}^r{}_r = 0$$

Consequently, these differentials are

$$de_\theta = e_\theta \frac{1}{r} dr - e_r r d\theta$$

$$de_r = e_\theta \frac{1}{r} d\theta$$

Determination of the Covariant Derivatives Relative to the Normalized Basis Vectors

GOTO page 31-13

The (algebraic) determination of $\nabla_{e_{\hat{k}}} e_{\hat{i}}$ and $\Gamma^{\hat{j}}_{\hat{i}\hat{k}}$ relative to the normalized basis acts as a check on the one relative to the coordinate basis

For example

$$\nabla_{e_{\hat{r}}} e_{\hat{\theta}} = \nabla_{e_r}\left(\frac{1}{r} e_\theta\right) = \nabla_{e_r}\left(\frac{1}{r}\right) e_\theta + \frac{1}{r} \nabla_{e_r} e_\theta$$

$$= -\frac{1}{r^2} e_\theta + \frac{1}{r}\cdot\frac{1}{r} e_\theta$$

$$= 0$$

$$\nabla_{e_{\hat{r}}} e_{\hat{r}} = \nabla_{e_r} e_r$$

$$= 0$$

These results agree with the behavior of $e_{\hat{\theta}}$ and $e_{\hat{r}}$ given in Figure 2 on page 31.2

Thus one has

$$\boxed{\begin{array}{l} \Gamma^{\hat{\theta}}_{\hat{\theta}\hat{r}} = 0 \ ; \ \Gamma^{\hat{r}}_{\hat{\theta}\hat{r}} = 0 \\ \Gamma^{\hat{\theta}}_{\hat{r}\hat{r}} = 0 \ ; \ \Gamma^{\hat{r}}_{\hat{r}\hat{r}} = 0 \end{array}}$$

These coefficient express the parallelism of the normalized under motion into the radial direction.

Similarly one has

$$\nabla_{e_{\hat\theta}} e_{\hat\theta} = \nabla_{\frac{1}{r}e_\theta}\left(\frac{1}{r}e_\theta\right) = \frac{1}{r}\nabla_{e_\theta}\left(\frac{1}{r}e_\theta\right)$$

$$= \frac{1}{r^2}\nabla_{e_\theta} e_\theta$$

$$= \frac{1}{r^2}(-)r\,e_r = \frac{-1}{r}e_{\hat r}$$

$$\nabla_{e_{\hat\theta}} e_{\hat r} = \frac{1}{r}\nabla_{e_\theta} e_r$$

$$= \frac{1}{r}\,\frac{1}{r}e_\theta$$

$$= \frac{1}{r}e_{\hat\theta}$$

Thus one has

$$\boxed{\begin{array}{ll} \Gamma^{\hat\theta}{}_{\hat\theta\hat\theta} = 0 \;; & \Gamma^{\hat r}{}_{\hat\theta\hat\theta} = -\frac{1}{r}\,\text{XX} \\ \Gamma^{\hat\theta}{}_{\hat r\hat\theta} = \frac{1}{r}\,\text{XX} & \Gamma^{\hat r}{}_{\hat r\hat\theta} = 0 \end{array}} \quad (**)$$

These quantities are also called <u>rotation coefficients</u> because they generate rotations of the normalized basis vectors under motion into the angular direction.

of the normaliz basis vectors $e_{\hat{j}}$

The differentials $d\underset{\sim}{e}_{\hat{i}} = e_{\hat{j}}\, \Gamma^{\hat{j}}{}_{\hat{i}\hat{k}}\, \underset{\sim}{\omega}^{\hat{k}}$ are (vector valued) linear combinations of the dual basis covectors $\underset{\sim}{\omega}^{\hat{\theta}}$ and $\underset{\sim}{\omega}^{\hat{r}}$.

They form the basis dual to the normalized basis $\left\{ e_{\hat{r}} = \frac{\partial}{\partial r},\ e_{\hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta} \right\}$ namely,

$\left\{ \underset{\sim}{\omega}^{\hat{r}} = dr,\ \underset{\sim}{\omega}^{\hat{\theta}} = r\,d\theta \right\}$

In terms of its elements one has with the help of Eq.(**) on P 3-10,

$$d e_{\hat{\theta}} = -e_{\hat{r}} \frac{1}{r} \omega^{\hat{\theta}} = -e_{\hat{r}}\, \underset{\sim}{d\theta}$$

$$d e_{\hat{r}} = e_{\hat{\theta}} \frac{1}{r} \omega^{\hat{\theta}} = e_{\hat{\theta}}\, \underset{\sim}{d\theta}$$

This result is consistent with the given relations, Eqs.(**) on page 31.14, as it must.