LECTURE 33

I. Cartan's 1st Structural Equation

II. The 1-2 version of Stokes Theorem

In Section §14.5 of MTW read pages 348-352.
MTW's Eqs. (14.11)-(14.13) as well as (14.14)-(14.16) are familiar from Lectures 29 and 30.
Eq. (14.21) and Exercise 14.6 on page 359 refer to the infinitesimal version of the 1-2 version of Stokes' Theorem $\iint_{\mathscr{D}} d\omega = \int_{\partial\mathscr{D}} \omega$.

33.2

Reminder

Every system of parallelism on a manifold has at each point P a geometrical footprint in the form of its torsion tensor

$$\mathbb{T}: T_P(M) \times T_P(M) \longrightarrow T_P(M)$$
$$(u, v) \rightsquigarrow \mathbb{T}(u,v) = \nabla_u v - \nabla_v u - [u,v]$$

It is a <u>geometric object</u> in that it is independent of one's choice of basis for the tangent space.

These geometric footprints of systems of parallel transport are concretized by the types of strain deformations exhibited by a crystal.

Let u and v be two lattice vectors which span a parallelogram when the crystal is in its unstrained state as depicted in Figure 33.1 a and in a strained as depicted in Figure 33.1 b.

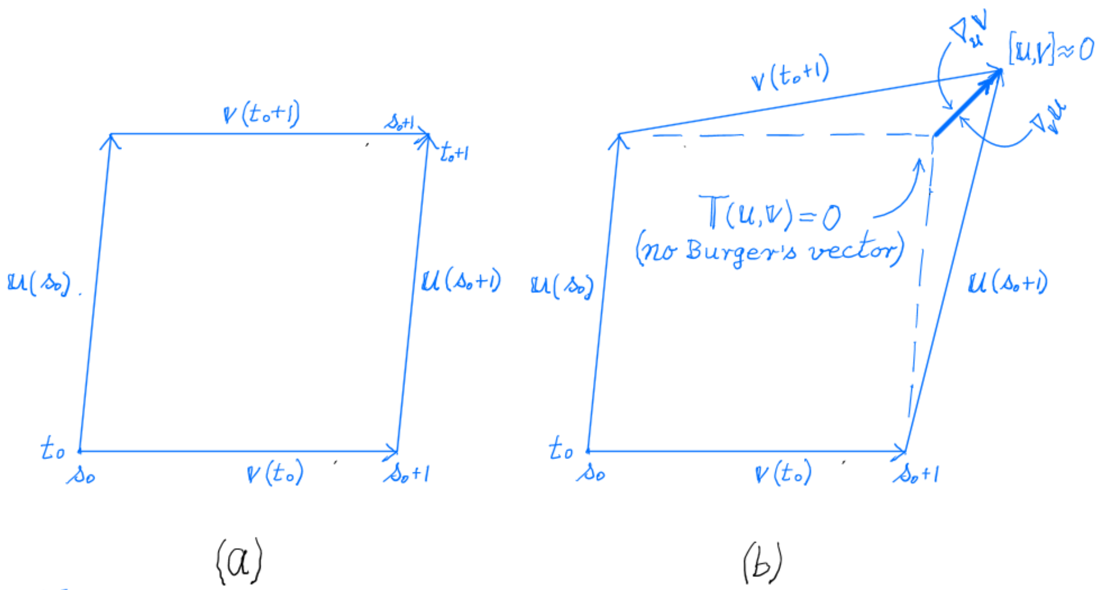

Figure 33.1: For an unstrained crystal without any

dislocations, two lattice vectors as depicted in panel (a) form a closed figure with two pairs of parallel sides, a parallelogram. By contrast that crystal in a strained state has non-parallel sides. As depicted in panel (b), the deviation from parallism is mathematized by $\nabla_u v = \nabla_v u$. (33.3)

By contrast, if the crystal is permeated by a distribution of dislocations, $\mathbb{T}(u,v) \neq 0$, even in its unstrained state as depicted in Figure 33.2 a.

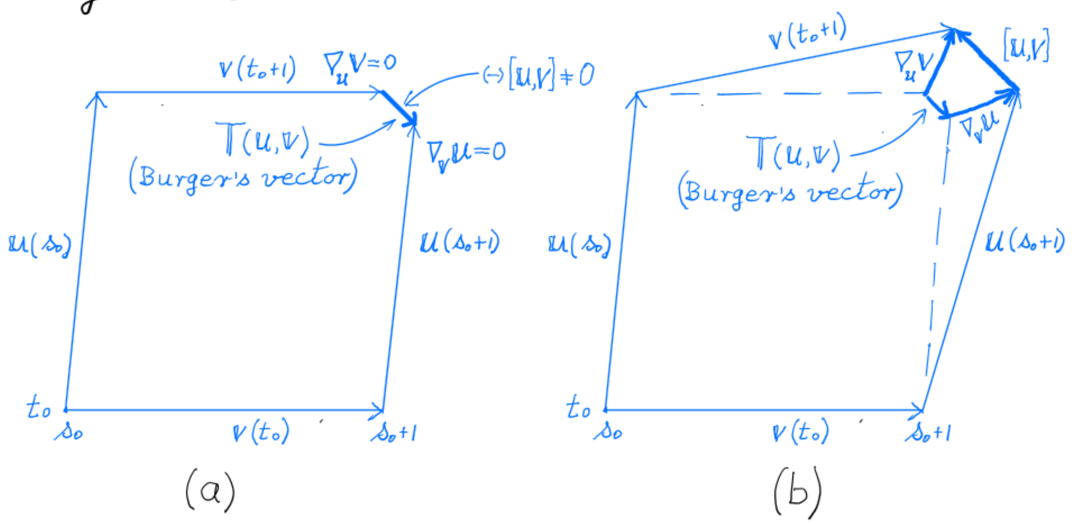

Figure 33.2: A crystal permeated by a distribution of dislocations has a non-zero Burger's vector, $\mathbb{T}(u,v) = \nabla_u v - \nabla_v u - [u,v] \neq 0$. It is non-zero both in its unstrained state ($\nabla_u v = \nabla_v u = 0$), as in panel (a), and in the case when the strain is non-zero ($\nabla_u v \neq 0, \nabla_v u \neq 0$), as in panel (b)

(33.4)

I. Cartan's 1st Structural Equation

The mapping

$$\mathbb{T}:\ T_p(M)\times T_p(M)\longrightarrow T_p(M)$$

$$(u,v)\ \rightsquigarrow\ \mathbb{T}(u,v)=\nabla_u v-\nabla_v u-[u,v] \qquad (33.1)$$

is pointwise linear, and hence is a tensor map, i.e multilinear map, a tensor, for each tangent space $T_p(M)$.

Being a tensor map,

$$\mathbb{T}(u,v)=\nabla_u v-\nabla_v u-[u,v]$$

can be expanded in terms of the vector basis $\{e_k\}$ and its dual basis $\{\omega^\ell\}$:

$$\begin{aligned}\mathbb{T}(u,v)&=\mathbb{T}(e_m u^m, e_n v^n)\\ &=\mathbb{T}(e_m,e_n)\,u^m v^n\\ &=e_k T^k{}_{mn}\langle\omega^m|u\rangle\langle\omega^n|v\rangle\\ &=e_k T^k{}_{mn}\ \omega^m\otimes\omega^n\ (u,v)\end{aligned}$$

This holds for all vectors u and v. Consequently,

$$\mathbb{T}=e_k T^k{}_{mn}\ \omega^m\otimes\omega^n \qquad (33.2)$$

This is a tensor of rank $\binom{1}{2}$. Its highlighted by Eqs. (15.1) and (16.1), its components $T^k{}_{mn}$ relative to the tensor basis $\{e_k\otimes\omega^m\otimes\omega^n\}$ are well-determined and unique.

The fact that

$$\begin{aligned}\nabla_u e_i&\equiv\langle de_i|u\rangle\\ &=e_j\langle\underset{\sim}{\omega}^j{}_i|u\rangle,\end{aligned}$$

together with Eqs. (33.1) and (33.2) imply that $\mathbb{T}$ is a vectorial two-form

whose structure is determined by the array of connection (33.5)
one-forms $\underset{\sim}{\omega}^j{}_i$ and the basis elements $\underset{\sim}{\omega}^i$. To that end express the vectors $\mathbb{u}$ and $\mathbb{v}$ in $\mathbb{T}(\mathbb{u},\mathbb{v})$ in terms of the basis elements $\mathbb{e}_i$ and their duals $\underset{\sim}{\omega}^j$:

$$\mathbb{u} = \mathbb{e}_i \langle \underset{\sim}{\omega}^i | \mathbb{u} \rangle = \langle \mathbb{e}_i \otimes \omega^i | \mathbb{u} \rangle \equiv \langle d\mathcal{P} | \mathbb{u} \rangle$$

$$\mathbb{v} = \mathbb{e}_j \langle \underset{\sim}{\omega}^j | \mathbb{v} \rangle = \langle \mathbb{e}_j \otimes \omega^j | \mathbb{v} \rangle \equiv \langle d\mathcal{P} | \mathbb{v} \rangle .$$

Introduce them into Eq. (33.1) and apply the product rule. The result is

$$\begin{aligned} \mathbb{T}(\mathbb{u},\mathbb{v}) \equiv \quad & \nabla_{\mathbb{u}} \mathbb{v} && - \nabla_{\mathbb{v}} \mathbb{u} && - [\mathbb{u},\mathbb{v}] \\ = \quad & \mathbb{e}_j \nabla_{\mathbb{u}} \langle \omega^j | \mathbb{v} \rangle && - \mathbb{e}_i \nabla_{\mathbb{v}} \langle \omega^i | \mathbb{u} \rangle && - \mathbb{e}_i \langle \omega^i | [\mathbb{u},\mathbb{v}] \\ & + (\nabla_{\mathbb{u}} \mathbb{e}_j) \langle \omega^j | \mathbb{v} \rangle && - (\nabla_{\mathbb{v}} \mathbb{e}_i) \langle \omega^i | \mathbb{u} \rangle && \end{aligned} \tag{33.3}$$

The first two terms are merely the directional derivatives of scalars

$$\nabla_{\mathbb{u}} \langle \omega^j | \mathbb{v} \rangle = \mathbb{u} (\omega^j | \mathbb{v}) \qquad (\equiv D_{\mathbb{u}} \langle \omega^j | \mathbb{v} \rangle)$$

$$\nabla_{\mathbb{v}} \langle \omega^i | \mathbb{u} \rangle = \mathbb{v} (\omega^i | \mathbb{u}) \qquad (\equiv D_{\mathbb{v}} \langle \omega^i | \mathbb{u} \rangle)$$

The last two terms with their covariant derivatives are linear combinations of the vectorial deviations,

$$\nabla_{\mathbb{u}} \mathbb{e}_j \equiv \mathbb{e}_i \langle \omega^i{}_j | \mathbb{u} \rangle \quad \text{and} \quad \nabla_{\mathbb{v}} \mathbb{e}_i \equiv \mathbb{e}_j \langle \omega^j{}_i | \mathbb{v} \rangle ,$$

of $\mathbb{e}_j$ and $\mathbb{e}_i$ away from parallelism, whose essence is mathematized by the connection 1-forms $\omega^i{}_j$. Introduce them into their difference and find that

$$\begin{aligned} (\nabla_{\mathbb{u}} \mathbb{e}_j) \langle \omega^j | \mathbb{v} \rangle - (\nabla_{\mathbb{v}} \mathbb{e}_i) \langle \omega^i | \mathbb{u} \rangle &= \mathbb{e}_i \left[\langle \omega^i{}_j | \mathbb{u} \rangle \langle \omega^j | \mathbb{v} \rangle - \langle \omega^j | \mathbb{u} \rangle \langle \omega^i{}_j | \mathbb{v} \rangle \right] \\ &= \mathbb{e}_i \left[\omega^i{}_j \otimes \omega^j - \omega^j \otimes \omega^i{}_j \right] (\mathbb{u},\mathbb{v}) \\ &\equiv \mathbb{e}_i \; \omega^i{}_j \wedge \omega^j \; (\mathbb{u},\mathbb{v}) \end{aligned}$$

The total expression for $\mathbb{T}(\mathbb{u},\mathbb{v})$ is therefore

$$\mathbb{T}(\mathbb{u},\mathbb{v}) = \mathbb{e}_i \Big[\mathbb{u} \big(\langle \omega^i | \mathbb{v} \rangle \big) - \mathbb{v} \big(\langle \omega^i | \mathbb{u} \rangle \big) - \langle \omega^i | [\mathbb{u},\mathbb{v}] \rangle + \omega^i{}_j \wedge \omega^j (\mathbb{u},\mathbb{v}) \Big].$$

33.6

Apply a combination of algebraic and calculus reasoning to the first three terms of this expression. The result is

$$\mathbb{U}(\langle \omega^i | \mathbb{V} \rangle) - \mathbb{V}(\langle \omega^i | \mathbb{U} \rangle) - \langle \omega^i | [\mathbb{U}, \mathbb{V}] \rangle \equiv d\omega^i(\mathbb{U}, \mathbb{V}), \qquad (33.4)$$

where the exterior derivative of a general 1-form $\underset{\sim}{\omega} = f\,dg$ is defined by

$$\begin{aligned} d\omega(\mathbb{U},\mathbb{V}) &= d(f\,dg)(\mathbb{U},\mathbb{V}) \\ &= df \wedge dg(\mathbb{U},\mathbb{V}) \\ &= df \otimes dg(\mathbb{U},\mathbb{V}) - dg \otimes df(\mathbb{U},\mathbb{V}) \\ &= \mathbb{U}(f)\,\mathbb{V}(g) - \mathbb{U}(g)\,\mathbb{V}(f) \end{aligned}$$

As we shall see on pages following pages 33.5, Eq. (33.4) is the 1-2 version of Stokes' theorem applied to an infinitesimal loop enclosing an infinitesimal area.

Introduce Stokes' theorem into the $\mathbb{T}(\mathbb{U},\mathbb{V})$. The result is

$$\mathbb{T}(\mathbb{U},\mathbb{V}) = \mathbb{e}_i \otimes \left[d\omega^i + \omega^i{}_j \wedge \omega^j \right](\mathbb{U},\mathbb{V}).$$

This holds for all vector fields $\mathbb{U}$ and $\mathbb{V}$. Thus one has the vector-valued 2-form

$$\boxed{\mathbb{T} = \mathbb{e}_i \otimes \left(d\underset{\sim}{\omega}^i + \underset{\sim}{\omega}^i{}_j \wedge \underset{\sim}{\omega}^j \right) \equiv \mathbb{e}_i \otimes \Omega^i} \qquad (33.5)$$

It is a tensor of rank $\binom{1}{2}$

The boxed equation is Cartan's 1st structural equation. It is an an explicit expression for the tensor map $\mathbb{T}$, <u>Cartan's torsion tensor</u>

$$\mathbb{T} = \mathbb{e}_i \otimes \underset{\sim}{\Omega}^i.$$

The 2-forms

$$\underset{\sim}{\Omega}^i = d\omega^i + \omega^i{}_j \wedge \omega^j \qquad i = 1, \cdots, n$$

are collectively referred to as Cartan's torsion 2-form.

By expanding $d\omega^i$ and $\omega^i{}_j \wedge \omega^j$ in terms of $\omega^k \wedge \omega^\ell$ one obtains

33.7

$$\mathbb{T} = \frac{1}{2!} \mathbf{e}_i T^i{}_{k\ell}\, \omega^k \wedge \omega^\ell$$
$$= \mathbf{e}_i T^i{}_{|k\ell|}\, \omega^k \wedge \omega^\ell$$

The parallel transport is said to be integrable when its torsion tensor vanishes,

$$\mathbf{e}_i \otimes \underset{\sim}{\Omega}^i = 0.$$

II. Stokes' Infinitesimal 1-2 Theorem

Consider the line integral $\oint \omega$ of an arbitrary 1-form, such as $\omega = f\,dg$, around a closed loop formed by two vector fields u and v:

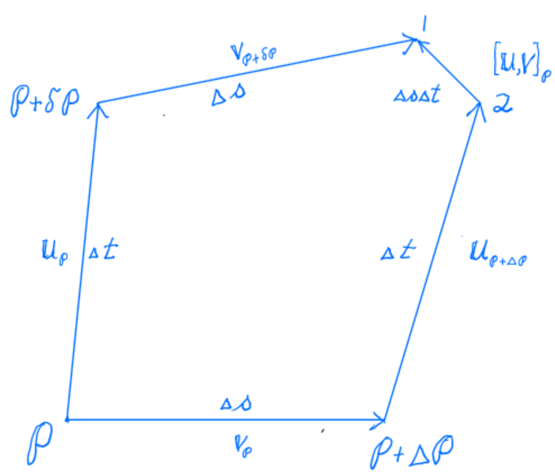

Figure 33.3: The vector fields u and v determine the integration domain for the infinitesimal version of Stokes' 1-2 theorem $\oint \omega = \iint d\omega$

$$\oint \omega = \int_{P}^{P+\delta P} \langle \omega | \mathbb{u} \rangle dt + \int_{P+\delta P}^{1} \langle \omega | \mathbb{v} \rangle ds + \int_{1}^{2} \langle \omega | (-)[\mathbb{u},\mathbb{v}] \rangle d(ts) + \int_{2}^{P+\Delta P} \langle \omega | (-)\mathbb{u} \rangle dt + \int_{P+\Delta P}^{P} \langle \omega | (-)\mathbb{v} \rangle ds \qquad (33.8)$$

Apply the mean value theorem for integrals whose limit are close together, and find to second order accuracy the closed line integral is

$$\oint \omega = \underbrace{\langle \omega | \mathbb{u} \rangle \Big|_{P} \Delta t - \langle \omega | \mathbb{u} \rangle \Big|_{P+\Delta P} \Delta t}_{-\Delta s \mathbb{v}(\langle \omega | \mathbb{u} \rangle \Delta t)} + \underbrace{\langle \omega | \mathbb{v} \rangle \Big|_{P+\delta P} \Delta s - \langle \omega | \mathbb{v} \rangle \Big|_{P} \Delta s}_{\Delta t \mathbb{u}(\langle \omega | \mathbb{v} \rangle \Delta s)} - \langle \omega | [\mathbb{u},\mathbb{v}] \Delta t \Delta s$$

$$= \Big\{ \mathbb{u}(\langle \omega | \mathbb{v} \rangle) - \mathbb{v}(\langle \omega | \mathbb{u} \rangle - \langle \omega | [\mathbb{u},\mathbb{v}] \rangle \Big\}_{P} \Delta t \Delta s$$

Without loss of generality, let $\underset{\sim}{\omega} = f\,dg$. Reasoning from calculus and linear algebra leads to the conclusion that the content of the curly braces condenses into the value $d\omega(\mathbb{u},\mathbb{v})$ of the exterior derivative of ω:

$$\{\mathbb{u}(\langle \omega | \mathbb{v} \rangle) - \mathbb{v}(\langle \omega | \mathbb{u} \rangle - \langle \omega | [\mathbb{u},\mathbb{v}] \rangle\}_{P} = d\omega(\mathbb{u},\mathbb{v})\Big|_{P}.$$

It follows that the value of the closed line integral is

$$\boxed{\oint \omega = d\omega(\mathbb{u},\mathbb{v})\Big|_{P} \Delta t \Delta s} \qquad (33.6)$$

By contrast, the surface integral over the area spanned by the displacement vectors $\Delta t \mathbb{u}$ and $\Delta s \mathbb{v}$ is the double integral

$$\oiint d\omega = \int_0^{\Delta t}\int_0^{\Delta s} d\omega(\mathbb{u},\mathbb{v})\, dt\, ds.$$

The integral over the small area enclosed by the small curve displacements is

$$\boxed{\oiint d\omega = d\omega(\mathbb{u},\mathbb{v})\Big|_{P} \Delta t \Delta s} \qquad (33.7)$$

Compare this surface integral with the value of the line integral, Eq. (33.6), and find that (33.9)

$$\boxed{\oint \omega = \oiint d\omega}$$

This is the 1-2 Stokes' theorem. By making measurements on the boundary of a domain one obtains properties that prevail in its interior.

Comments

Consider the 2-d surface

$$x^k(s,t)\ ;\ k=1,\dots,n \quad \begin{cases} 0 \le s \le s_1 \\ 0 \le t \le t_1 \end{cases}$$

with its two tangent vector fields

$$u = \frac{\partial x^k(s,t)}{\partial t}\frac{\partial}{\partial x^k} \equiv U^k(s,t)\frac{\partial}{\partial x^k}$$

$$v = \frac{\partial x^\ell(s,t)}{\partial s}\frac{\partial}{\partial x^\ell} \equiv v^\ell(s,t)\frac{\partial}{\partial x^\ell}$$

A.) Let $\underset{\sim}{\omega} = f(x)\, dg(x)$ be a given 1-form.
Then its line integral $\int_{\mathcal{P}}^{\mathcal{P}+\delta\mathcal{P}} \omega$ along the integral curve of u passing through $\mathcal{P}$ and $\delta\mathcal{P}$ (with s held fixed at $s=a$) is evaluated as follows:

$$\int_{\mathcal{P}}^{\mathcal{P}+\delta\mathcal{P}} \omega = \int_{t=0}^{t=\Delta t} \langle f\, dg \,|\, u\rangle\, dt$$

$$\equiv \int_0^{\Delta t} f\left(x^i(s,t)\right)\frac{\partial g(x^i(s,t))}{\partial x^k}\, u^k(s,t)\bigg|_{s=a} dt$$

B.) Let $d\omega = df \wedge dg$ be the exterior derivative of ω.
Then the surface integral of $d\omega$ over the given surface is evaluated as follows:

$$\oiint df \wedge dg \equiv \int_0^{t_1}\int_0^{s_1} df \wedge dg\,(u,v)\, dt\, ds$$

33.10

$$= \int_0^{t_1}\int_0^{s_1} [df\otimes dg - dg\otimes df]\left(u^k\frac{\partial}{\partial x^k}, v^\ell \frac{\partial}{\partial x^\ell}\right)$$

$$= \int_0^{t_1}\int_0^{s_1} \left[u^k \frac{\partial f}{\partial x^k} v^\ell \frac{\partial g}{\partial x^\ell} - u^k\frac{\partial g}{\partial x^k} v^\ell \frac{\partial g}{\partial x^\ell}\right]_{x^i(s,t)} dt\,ds$$

$$= \int_0^{t_1}\int_0^{s_1} \det\begin{vmatrix} u^k f_{,k} & v^\ell f_{,\ell} \\ u^k g_{,k} & v^\ell g_{,\ell}\end{vmatrix} dt\,ds$$

C.) Let $\underset{\sim}{\omega} = \omega_i\, \underset{\sim}{d}x^i$ and consider its exterior derivative

$$d\omega = d(\omega_i\, dx^i) = \frac{\partial \omega_i}{\partial x^j}\, dx^j \wedge dx^i$$

$$= \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i}\right) dx^i \otimes dx^j$$

Evaluate the surface integral $\iint_{\mathcal{D}} d\omega$ over the 2-d domain spanned by

$$\mathcal{D} = \{x^k(s,t) : k = 1,\ldots,n;\ 0\le s\le s_1;\ 0\le t\le t_1\}$$

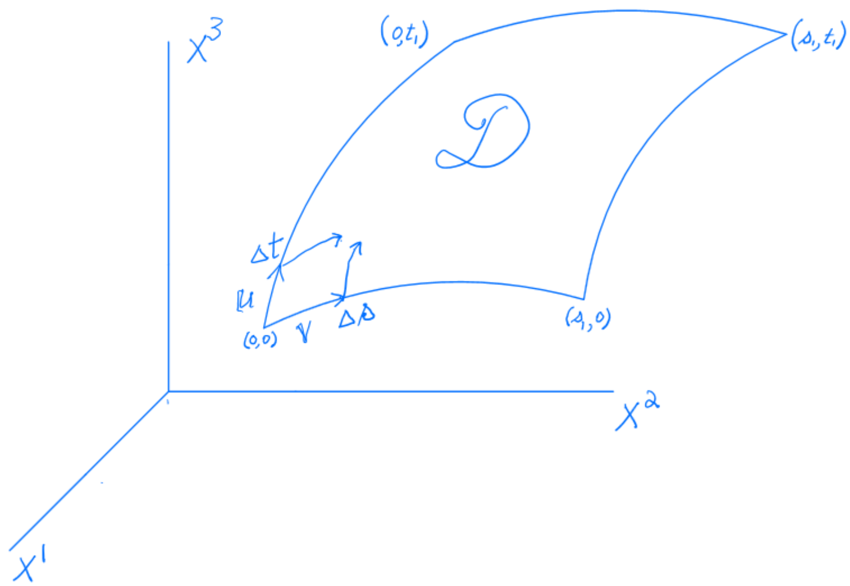

Figure 33.4: Two-dimensional integration domain $\mathcal{D}$ for the surface integral $\iint d\omega$.

33.11

$$\iint_{\mathcal{D}} d\omega = \iint_{\mathcal{D}} \frac{\partial \omega_i}{\partial x^j}\, dx^j \wedge dx^i$$

$$= \int_0^{t_1}\int_0^{s_1} \omega_{i,j}\; dx^j \wedge dx^i\,(\mathbb{u},\mathbb{v})\, dt\, ds$$

$$\doteq \iint \omega_{i,j}\; \underbrace{\underbrace{dx^j \wedge dx^i \left(\frac{\partial x^k}{\partial t}\frac{\partial}{\partial x^k}, \frac{\partial x^\ell}{\partial s}\frac{\partial}{\partial x^\ell}\right)}_{\left[\delta^j_k \delta^i_\ell - \delta^j_\ell \delta^i_k\right]\frac{\partial x^k}{\partial t}\frac{\partial x^\ell}{\partial s}}}_{\det\left|\frac{\partial(x^j, x^i)}{\partial(t,s)}\right|}\; dt\, ds$$

Thus,

$$\iint_{\mathcal{D}} d\omega = \int_0^{t_1}\int_0^{s_1} \omega_{i,j}\left(x^k(s,t)\right) \det\left|\frac{\partial(x^j,x^i)}{\partial(t,s)}\right| dt\, ds$$

For small $s_1=\Delta s$, $t_1=\Delta t$ one simply has

$$\iint_{\mathcal{D}} d\omega = d\omega(\mathbb{u},\mathbb{v})\,\Delta t\,\Delta s$$

$$= \omega_{i,j}\left(x^k(0,0)\right) \det \left.\frac{\partial(x^j,x^i)}{\partial(t,s)}\right|_{(0,0)} \Delta t\,\Delta s\ .$$