(34.1)

LECTURE 34

I. Curvature-induced rotation

II. Riemann curvature: Parallel transporting a vector around a loop.

Read Sections 11.4 and 8.7 in MTW

I. Curvature: Where does it come from?

A given law of parallel transport provides rules for constructing parallel vectors. The application of this construction to two vectors emanating from the same point yields a parallelogram. However, whether or not the parallelogram is a closed figure or not depends on the type of parallel transport as expressed by the torsion tensor

$$\mathbb{T} = \mathbf{e}_i \otimes (d\omega^i + \omega^i{}_j \wedge \omega^j) \equiv \mathbf{e}_i \underset{\sim}{\Omega}^i .$$

When evaluated on two vectors it yields a displacement vector. Thus parallel transport has a displacement aspect, and torsion, via parallelograms mathematizes this displacement into precise form.

But parallel transport has also a rotation aspect to it. This is revealed by means of the curvature via vectors parallel-transported around a closed loop.

EXAMPLE (Rotation of a Vector Transported around a Circular Loop)

To illustrate curvature-induced rotation, focus on parallel transport characterized by Schild's closed parallelogram construction on the two sphere S^2. Take a closed circle of constant latitude θ, and using the closed parallelogram construction, parallel transport a vector around this loop.

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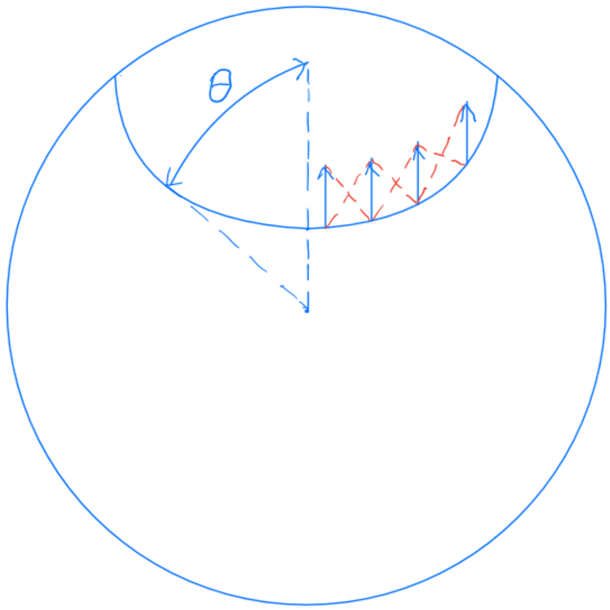

Figure 34.1: Vector parallel-transported around a latitude circle.

The vectors are parallel because they are opposite sides of a succession of parallelograms.
The parallelism of these opposing sides is inherited from the parallelism of the ambient 3-d Euclidean space. It induces a unique parallel transport on the two-sphere S^2.

The result of this parallel transport around the closed circle of constant latitude is a product of 3-d and 2-d geometrical reasoning:
(i) Cut out an annulus strip surrounding the circle of constant latitude,

$$\theta = \text{constant}$$

This strip is tangent to the bottom of cone whose apex angle is (34.4)

$$2 \times (90^\circ - \theta)$$

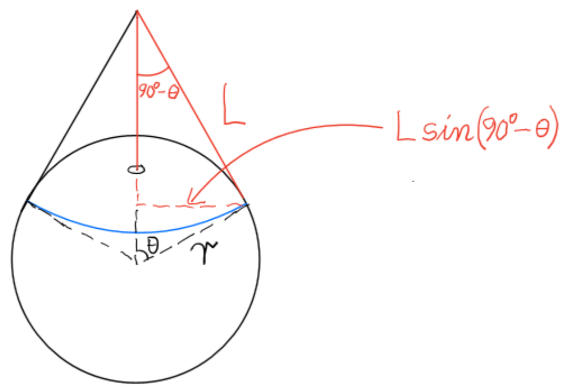

Figure 34.2: Cone on top a sphere of radius r. The two are tangent along the circle of constant latitude $\theta = \text{const.}$ The circumference of the circle of constant latitude is $2\pi L \sin(90^\circ - \theta)$.

(ii) Cut this cone along an edge and flaten the cone so that it becomes a disk with a missing angle α.

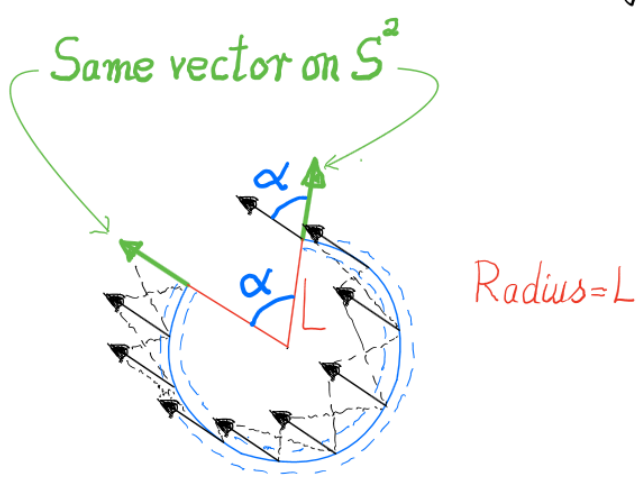

Figure 34.3: Vectors parallel on the strip of tangency. Its length is the circumference of the circle of constant latitude, $2\pi L \sin(90° - \theta)$. The ends of the strip together with their (green) reference vectors are to be identified as one and the same.

The strip of tangency supports the parallel vectors attached to the spread out circular arc. Its length equals the circumference of the latitude circle, namely $2\pi L \sin(90° - \theta)$.

(iii) The vectors emanating from the spread out arc on the tangency strip in Figure 34.2 are parallel. This fact guarantees that, upon completing its parallel transport around the latitude loop, a vector will rotate exactly by the angular amount α that is missing from the spread out circular arc.

More precisely, on has

α = amount of rotation upon parallel transport around a closed loop;

α = rotation (angle) of a vector after parallel transport around a circle at latitude θ away from the North pole;

$$\alpha = \frac{\begin{pmatrix}\text{arclength of} \\ \underline{\text{completed}}\text{ circle}\end{pmatrix} - \begin{pmatrix}\text{arclength of} \\ \underline{\text{latitude}}\text{ circle}\end{pmatrix}}{L}$$

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$$\alpha = \frac{2\pi L - 2\pi L \sin(90° - \theta)}{L}$$

$$\boxed{\alpha = 2\pi(1 - \cos\theta)}$$

This rotation angle lead to the following heuristic Definition ("Curvature")

$\alpha \equiv (\text{curvature}) \times \text{enclosed area};$

or

$$\text{curvature} \equiv \frac{\text{rotation angle}}{\text{enclosed area}} = \frac{\alpha}{\int_0^{2\pi} d\varphi \int_0^{\theta} r^2 \sin\theta \, d\theta} = \frac{2\pi(1-\cos\theta)}{2\pi r^2 (1-\cos\theta)} = \frac{1}{r^2}.$$

Remark

The radius r is called the <u>radius of curvature</u> of the enclosed area.

II. Curvature as the generator of rotation under parallel transport around a closed curve.

The line of reasoning leading to this conclusion is as follows:

A) Given: An area enclosed by a closed curve.

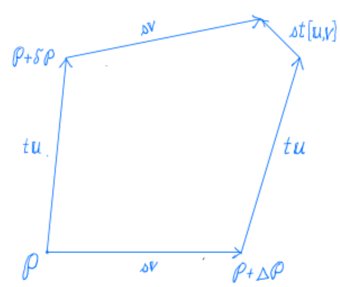

Figure 33.4: A closed curve as determined by two vector fields and their commutator

B)(i) On this closed curve introduce an arbitrarily specified "fiducial" vector field w.

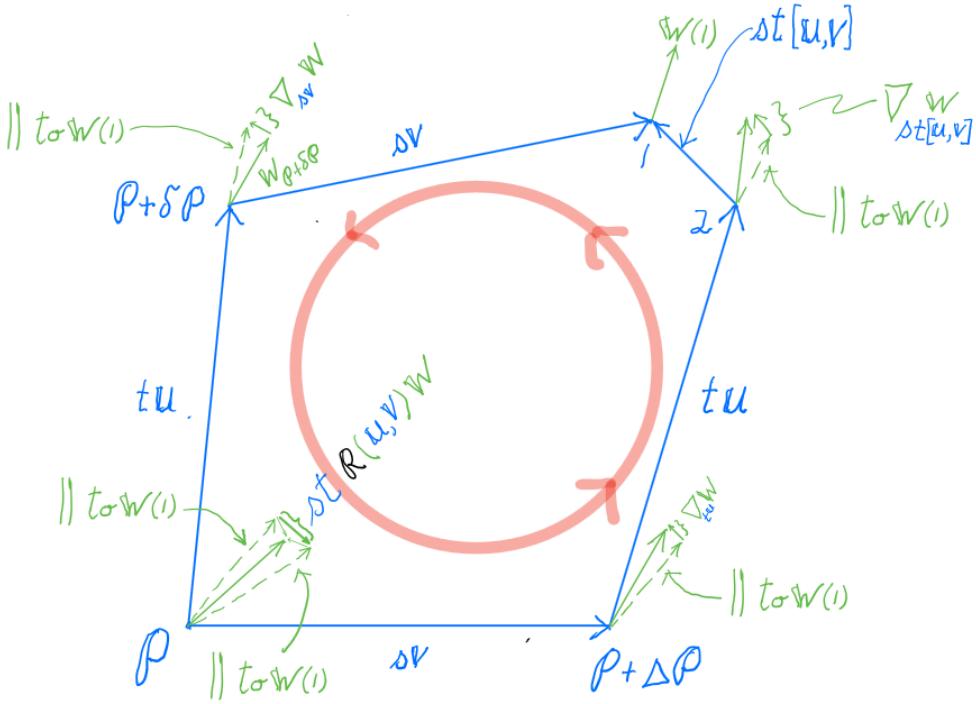

Figure 34.5: Parallel transport vector w around a closed loop.

(ii) Parallel transport the vector $w(1)$ along two different broken paths:

$$1 \longrightarrow \mathcal{P}+\delta\mathcal{P} \longrightarrow \mathcal{P}$$

and

$$1 \longrightarrow 2 \longrightarrow \mathcal{P}+\Delta\mathcal{P} \longrightarrow \mathcal{P}\,.$$

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C) Parallel transport $W(1)$ along $1 \longrightarrow P+\delta P \longrightarrow P$ by means of a Taylor series along each segment to 2^{nd} order accuracy:

① $\begin{pmatrix}\text{vector at } P+\delta P \text{ which} \\ \text{is parallel to } W(1) \\ \text{and has been} \\ \text{parallel tranported} \\ \text{from } 1 \text{ to } P+\delta P\end{pmatrix} = W\big|_{P+\delta P} + s\nabla_V W\big|_{P+\delta P} + \frac{s^2}{2!}\nabla_V\nabla_V W\big|_{P+\delta P}$ + higher order terms

② The parallel translate to point P of the vector in ① is

$$\underbrace{W_P + t\nabla_U W\big|_P + \frac{t^2}{2!}\nabla_U\nabla_U W + \cdots}_{\text{parallel translate of } W_{P+\delta P}} + \underbrace{s\nabla_V W\big|_P + ts\nabla_U\nabla_V W\big|_P + \cdots}_{\text{parallel translate of } s\nabla_V W|_{P+\delta P}} + \frac{s^2}{2}\nabla_V\nabla_V W\big|_P + \cdots$$

$= \begin{pmatrix}\text{vector at } P \text{ which is parallel to } W(1) \text{ and has been parallel} \\ \text{translated along } 1 \longrightarrow P+\delta P \longrightarrow P\end{pmatrix}$

D) Parallel transport of $W(1)$ along $1 \longrightarrow 2 \longrightarrow P+\Delta P \longrightarrow P$ yields

(same as C)② except $sV \leftrightarrow tU$ $+ st\,\nabla_{[U,V]}W\big|_P =$

$=$ parallel translate of $W(1)$ along $1 \longrightarrow 2 \longrightarrow P+\Delta P \longrightarrow P$

E) (Parallel translate of $W(1)$ to point P along $1 \longrightarrow P+\delta P \longrightarrow P$)

$-$ (Parallel translate of $W(1)$ to point P along $1 \longrightarrow 2 \longrightarrow P+\Delta P \longrightarrow P$) $=$

$=$ vector in C)② $-$ vector in D)

$$= st\left[\nabla_U\nabla_V - \nabla_V\nabla_U - \nabla_{[U,V]}\right]W + \cdots$$

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$\equiv \mathcal{A}t\, \mathcal{R}(\mathbb{u},\mathbb{v})\,\mathbb{w}$

$=$ "vectorial amount of rotation that $\mathbb{w}_p$ undergoes when translated around the closed path counterclock wise."

The operator

$$\mathcal{R}: \quad T_p(M)\times T_p(M)\times T_p(M) \longrightarrow T_p(M)$$

$$(\mathbb{u},\mathbb{v},\mathbb{w}) \rightsquigarrow \mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{w} = \left(\nabla_{\mathbb{u}}\nabla_{\mathbb{v}} - \nabla_{\mathbb{v}}\nabla_{\mathbb{u}} - \nabla_{[\mathbb{u},\mathbb{v}]}\right)\mathbb{w}$$

is called the curvature operator.

This operator is a "tensor map", i.e. it is pointwise linear in in each of its arguments

$$\boxed{\begin{aligned} \mathcal{R}(f\mathbb{u},\mathbb{v})\mathbb{w} &= f\,\mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{w} \\ \mathcal{R}(\mathbb{u},g\mathbb{v})\mathbb{w} &= g\,\mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{w} \\ \mathcal{R}(\mathbb{u},\mathbb{v})h\mathbb{w} &= h\,\mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{w} \end{aligned}}$$

The components of this tensor map are those of a tensor of rank $\binom{1}{3}$. Indeed, letting

$$\mathbb{u} = e_i u^i$$

$$\mathbb{v} = e_j v^j$$

$$\mathbb{w} = e_k w^k$$

and using the fact $\mathcal{R}(\mathbb{u},\mathbb{v}) = -\mathcal{R}(\mathbb{v},\mathbb{u})$, one finds

$$\mathcal{R}(\mathbb{u},\mathbb{v})(\mathbb{w}) = \left[\mathcal{R}(e_i,e_j)e_k\, u^i v^j\right](\mathbb{w})$$

$$\equiv \left[e_\ell \otimes \omega^k \otimes R^\ell{}_{kij}\, \omega^i \wedge \omega^j / 2!\, (u,v) \right] (W) \qquad (34.1)$$

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The tensor product inside the square bracket

$$\boxed{e_\ell \otimes \omega^k \otimes \underset{\sim}{R}^\ell{}_k = e_\ell \otimes \omega^k \otimes R^\ell{}_{kij}\, \omega^i \wedge \omega^j / 2! \equiv \mathbb{R}} \qquad (34.2)$$

is a $\binom{1}{1}$ tensor-valued 2-form. It is the curvature two-form.
It is a tensor of rank $\binom{1}{3}$, the Riemann tensor $\mathbb{R}$, whose components are the coefficients $R^\ell{}_{kij} = -R^\ell{}_{kji}$.

COMMENT 1

The tensor-valued two-form, Eq. (34.2) is to be compared with other similar but different two-forms.

1.) The torsion two-form

$$e_\ell \underset{\sim}{T}^\ell = e_\ell T^\ell{}_{ij}\, \omega^i \wedge \omega^j / 2! = \mathbb{T} \qquad (34.3)$$

It is a vector-valued 2-form, a tensor of rank $\binom{1}{2}$, the torsion tensor $\mathbb{T}$. Its components are the coefficients $T^\ell{}_{ij} = -T^\ell{}_{ji}$.

2.) The electromagnetic 2-form

$$\underset{\sim}{F} = F_{\mu\nu}\, \omega^\mu \wedge \omega^\nu / 2! = \mathbf{F} \qquad (34.4)$$

It is a scalar (electromagnetic flux)-valued 2-form, a tensor of rank $\binom{0}{2}$, the Faraday tensor $\mathbf{F}$. Its components are the coefficients $F_{\mu\nu} = -F_{\nu\mu}$.

The common denominator of the 2-forms (34.2)-(34.4) is that their respective properties are attributes intrinsic to whatever 2-d surface area is the target of interest.

COMMENT 2

Curvature, as defined heuristically on page 34.6, is given by

$$\text{curvature} \equiv \frac{\text{rotation angle}}{\text{enclosed area}} . \qquad (34.5)$$

Or more briefly, curvature equals rotation per unit area. Its relation to $st\left[e_\ell \otimes \omega^k \otimes R^\ell{}_{kij}\, \omega^i \wedge \omega^j / 2!\,(u,v)\right](W)$ is $\left(\text{curvature} \cdot {\text{enclosed} \atop \text{area}}\right)(W) = \Delta W = \left({\text{rotation} \atop \text{angle}}\right)(W)$.

1.) In the framework of a given metric tensor field

$$g = g_{mn}\, \omega^m \otimes \omega^n, \quad \text{where } g_{mn} = e_m \cdot e_n \; (= g(e_m, e_n)),$$

the curvature components (as shown on page 37.3 in Lecture 37) are anti-symmetric in their first two indeces:

$$g_{m\ell} R^\ell{}_{kij} = R_{mkij} = -R_{kmij}$$

2.) Recall from Figure 34.5 that the effect of parallel transporting the vector W around the u-v determined loop is the vectorial change which is given by

$$\Delta W = st\left[e_\ell \otimes \omega^k \underset{\sim}{R}^\ell{}_k(u,v)\right](W)$$

$$= st\; e_\ell\, w^k \underset{\sim}{R}^\ell{}_k(u,v)$$

$$= st\; e_\ell\, w^k g_{km} \underset{\sim}{R}^{\ell m}(u,v)$$

$$= st\left[e_\ell \otimes e_m \underset{\sim}{R}^{\ell m}(u,v)\right] \cdot W \qquad R^{\ell m}{}_{ij} = -R^{m\ell}{}_{ij}$$

$$\Delta W = st\, \tfrac{1}{2} \underset{\sim}{R}^{\ell m}(u,v)\, e_\ell \wedge e_m \cdot W \qquad (34.6)$$

3.) The coefficient $st\,\underset{\sim}{R}^{\ell m}(u,v)$ of each term in this sum is the angular amount by which W gets rotated in the plane spanned by the vectors e_ℓ and e_m. Each term in the sum is

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the infinitesimal vectorial change suffered by W due to this rotation. The double sum in Eq.(34.6) is the total vectorial change in W due to the contribution from each elementary rotation in the $e_\ell - e_m$ plane.

4.) In the illustrative example of a two-sphere on page 34.2 there only a single plane of rotation, and the relation between Eq.(34.5) for the two-sphere S^2 and Eq.(34.6) for the general case is

$$\Delta W = \left(\text{curvature} \cdot \begin{matrix}\text{enclosed}\\ \text{area}\end{matrix}\right)(W) = \begin{pmatrix}\text{rotation}\\ \text{angle}\end{pmatrix}(W).$$