LECTURE 35

(35.1)

I. Cartan's 2^{nd} Structure Equation

II. Cartan's Unit-economical Derivation of his Two Structure Equations.

In MTW read P 348–354 (§ 14.5)

<u>Comment</u>:

A comparison between MTW's Eq.(14.39) and (14.20) will lead to confusion because their "$\mathcal{R}$" in Eq. (14.39) is a <u>matrix</u> of 2-forms. By contrast, their "$\mathcal{R}$" in Eq.(14.20), (15.12), (15.14), (15.16), (15.19)–(15.21), (15.25), (15.26) is a dyad (i.e. a $\binom{1}{1}$ or $\binom{2}{0}$ tensor)-valued 2-form. To avoid this confusion and to highlight the dyad-valued nature of "$\mathcal{R}$" in these equation, MTW should have replace that "$\mathcal{R}$" with $\overleftrightarrow{\mathcal{R}}$ instead.

35.2

I. Cartan's 2^{nd} Structure Equation

The Riemann curvature tensor is the point-wise linear map,

$$\mathbb{R} : T_{\wp}(M) \times T_{\wp}(M) \times T_{\wp}(M) \longrightarrow T_{\wp}(M),$$

$$(\mathbb{W}, \mathbb{u}, \mathbb{v}) \quad \rightsquigarrow \mathbb{R}(\cdots, \mathbb{W}, \mathbb{u}, \mathbb{v}) = \{\nabla_{\mathbb{u}}\nabla_{\mathbb{v}} - \nabla_{\mathbb{v}}\nabla_{\mathbb{u}} - \nabla_{[\mathbb{u},\mathbb{v}]}\}\mathbb{W} \equiv \overleftrightarrow{\mathbb{R}}(\mathbb{u}, \mathbb{v})\mathbb{W}$$

which produces an amount of curvature-induced vectorial rotation from a given vector that has been parallel transported around the perimeter of the area enclosed by the curve segments of two vector fields $\mathbb{u}$, $\mathbb{v}$, and their commutator $[\mathbb{u}, \mathbb{v}]$,

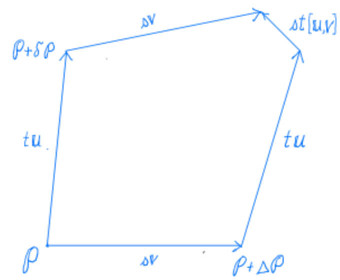

Figure 34.1 : A closed curve as determined by two vector fields and their commutator

Arrive at Cartan's 2^{nd} structure equation by using the same kind of algebraic and calculus reasoning as in his 1^{st} equation. Do this by using the modern post-WW II approach, i.e. Cartan's formulation of Stokes' theorem in terms of his exterior derivative applied to any diffential 1-form ω, namely

$$\mathbb{u}\,(\langle\omega\,|\,\mathbb{v}\rangle) - \mathbb{v}\,(\langle\omega\,|\,\mathbb{u}\rangle) - \langle\omega\,|\,[\mathbb{u},\mathbb{v}]\rangle = d\omega\,(\mathbb{u},\mathbb{v}).$$

Then obtain the structure equation in four steps:

STEP 1.

35.3

Focus separately on each of the three terms of the curvature-induced rotation mapping,

$$\{\underset{①}{\nabla_{u}\nabla_{v}} - \underset{②}{\nabla_{v}\nabla_{u}} - \underset{③}{\nabla_{[u,v]}}\}W = ① - ② - ③,$$

where

$$\left.\begin{aligned} ① &= \nabla_{u}\nabla_{v}\,(e_k w^k) \\ ② &= \nabla_{v}\nabla_{u}\,(e_k w^k) \\ ③ &= \nabla_{[u,v]}\,(e_k w^k) \end{aligned}\right\} \quad w^k = \langle \underset{\sim}{\omega}^k | W \rangle \ \text{ with } \langle \underset{\sim}{\omega}^k | e_\ell \rangle = \delta^k{}_\ell$$

STEP 2.

Using the Leibnitz product rule of the covariant derivative ∇, find that all 1st and 2nd derivative terms referring to $u(w^k)$, $v(w^k)$, $uv(w^k)$, and $vu(w^k)$ cancel.

STEP 3.

A) Introduce the Cartan connection 1-forms $\underset{\approx}{\omega}^j{}_k$,

$$\nabla_u(e_k) = e_\ell \langle \underset{\sim}{\omega}^\ell{}_k | u \rangle$$
$$\nabla_v(e_k) = e_\ell \langle \omega^\ell{}_k | v \rangle$$

into the remaining terms of ① - ② - ③. The result is

$$① - ② - ③ = e_\ell \Big[\overbrace{\langle \omega^\ell_j | u \rangle \langle \omega^j{}_k | v \rangle - \langle \omega^\ell{}_j | v \rangle \langle \omega^j{}_k | u \rangle}^{(\omega^\ell{}_j \otimes \omega^j{}_k - \omega^j{}_k \otimes \omega^\ell{}_j)(u,v) = \omega^\ell_j \wedge \omega^j_k (u,v)} \Big] w^k$$
$$+ e_\ell \Big[u(\langle \omega^\ell{}_k | v \rangle) - v(\langle \omega^\ell{}_k | u \rangle) - \langle \omega^\ell{}_k | [u,v] \rangle \Big] w^k$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{d\omega^\ell{}_k(\mathbb{u},\mathbb{v})}$$

35.4

The first two terms consolidate into the wedge product $e_\ell w^k \omega^\ell{}_j \wedge \omega^\ell{}_k$ evaluated on $(\mathbb{u},\mathbb{v})$.

Referring to Eq. (33.4) on page 33.6 of Lecture 33, note that the last three terms are proportional to $d\omega^\ell{}_k$, the exterior derivative of $\omega^\ell{}_k$ evaluated on $(\mathbb{u},\mathbb{v})$.

Thus, in ①-②-③ all 1st derivative terms referring to $\mathbb{u}(v^k)$ and $\mathbb{v}(u^k)$ cancel.

B.) The final (point-wise liner) result is

$$①-②-③ = e_\ell w^k [d\omega^\ell{}_k + \omega^\ell{}_j \wedge \omega^j{}_k](\mathbb{u},\mathbb{v})$$

or, in light of

$$w^k = \omega^k(\mathbb{w}),$$

$$①-②-③ = e_\ell \otimes \omega^k [d\omega^\ell{}_k + \omega^\ell{}_j \wedge \omega^j{}_k](\mathbb{w},\mathbb{u},\mathbb{v})$$

$$\equiv \mathbb{R}(\cdots;\mathbb{w},\mathbb{u},\mathbb{v})$$

Here

$$\mathbb{R} = e_\ell \otimes \omega^k \otimes [d\omega^\ell{}_k + \omega^\ell{}_j \wedge \omega^j{}_k]$$

the Riemann tensor. It is simply a tensor of rank $\binom{1}{3}$.

Being a multilinear map, the Riemann tensor $\mathbb{R}$ has well-determined components relative a chosen basis. They are

$$\mathbb{R}(\omega^r; e_s, e_m, e_n) \equiv R^r{}_{smn}$$

$$= \langle \omega^r | [\nabla_{e_m}\nabla_{e_n} - \nabla_{e_n}\nabla_{e_m} - \nabla_{[e_m,e_n]}] e_s \rangle$$

$$= (d\omega^r{}_s + \omega^r{}_j \wedge \omega^j{}_s)(e_m, e_n) \qquad (35.5)$$

STEP 4.

However, Cartan initiated a more penetrating and fruitful perspective on this multilinear map. His view is to rewrite this tensor in the form

$$\mathbb{R} = e_\ell \otimes \omega^k \underset{\sim}{\mathcal{R}}^\ell{}_k$$

where

$$\underset{\sim}{\mathcal{R}}^\ell{}_k \equiv \underset{\sim}{d}\underset{\sim}{\omega}^\ell{}_k + \underset{\sim}{\omega}^\ell{}_j \wedge \underset{\sim}{\omega}^j{}_k$$

and collectively call this the <u>curvature</u> <u>2-form</u>.

According to this perspective,

$$\mathbb{R} = \overleftrightarrow{\underset{\sim}{\mathcal{R}}} = e_\ell \otimes \omega^k \underset{\sim}{\mathcal{R}}^\ell{}_k$$

is a <u>tensor-valued</u> two-form. When it is evaluated on a pair of area spanning vectors u and v, the result is the tensor

$$\overleftrightarrow{\underset{\sim}{\mathcal{R}}}(u,v) = e_\ell \otimes \omega^k \underset{\sim}{\mathcal{R}}^\ell{}_k(u,v).$$

It is the generator of rotations of vectors parallel transported around the perimeter of the (curvature permeated) area spanned by u and v. If w is one of those vectors, then it will undergo the vectorial change

$$\overleftrightarrow{\mathcal{R}}(u,v)(w) = e_\ell\, w^k \underset{\sim}{\mathcal{R}}^\ell{}_k(u,v).$$

The infinitesimally rotated vector is

$$\left[\overleftrightarrow{I} + \overleftrightarrow{\mathcal{R}}(u,v)\right] w = e_\ell \left[\delta^\ell_k + \mathcal{R}^\ell{}_k(u,v)\right] w^k \equiv \text{r.h.s.}$$

(35.6)

By going from index notation to matrix notation the r.h.s. expression for the rotated vector is

$$\text{r.h.s.} = [\mathbf{e}_1 \quad \cdots \quad \mathbf{e}_n] \begin{bmatrix} \\ \delta^\ell_k + R^\ell_k(\mathbf{u},\mathbf{v}) \\ \\ \end{bmatrix} \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}$$

CONCLUSION

The tensor-valued curvature two-form

$$\mathbb{R} = \boxed{\begin{aligned} \overleftrightarrow{R} &= \mathbf{e}_k \otimes \omega^\ell \left(d\omega^k{}_\ell + \omega^k{}_j \wedge \omega^j{}_\ell \right) = \mathbf{e}_k \otimes \omega^\ell \; R^k{}_{\ell|mn|} \, \omega^m \wedge \omega^n \\ &= \mathbf{e}_k \otimes \omega^\ell \tfrac{1}{2} R^k{}_{\ell mn} \, \omega^m \wedge \omega^n \end{aligned}}$$

is Cartan's second structure equation.

His first structure equation is the vector-valued torsion two-form

$$\mathbb{T} = \boxed{\begin{aligned} \vec{T} &= \mathbf{e}_k \left(d\omega^k + \omega^k{}_j \wedge \omega^j \right) = \mathbf{e}_k \; T^k{}_{|mn|} \, \omega^m \wedge \omega^n \\ &= \mathbf{e}_k \tfrac{1}{2} T^k{}_{mn} \, \omega^m \wedge \omega^n \end{aligned}}$$

(35.7)

II. The 1st and 2nd structure equation via Cartan's line of reasoning

He obtained his two equations in a unit-economical way by applying the exterior derivative to

1.) his unit tensor

$$\boxed{\mathbf{e}_i \otimes \omega^i \equiv d\mathcal{P}} \qquad (35.1)$$

and

2.) his mathematization parallel transport

$$\boxed{d\mathbf{e}_i = \mathbf{e}_j \otimes \omega^j{}_i} \qquad (35.2)$$

He obtains the 1st structure equation as follows:

Take the exterior derivative of Eq. (35.1) and use Eq. (35.2):

$$d(d\mathcal{P}) = d(\mathbf{e}_i \otimes \omega^i)$$

$$= \mathbf{e}_i \otimes d\omega^i + d\mathbf{e}_i \wedge \omega^i$$

$$\boxed{d^2\mathcal{P} =} \boxed{\mathbf{e}_j \otimes (d\omega^j + \omega^j{}_i \wedge \omega^i) = \mathbf{e}_j T^j{}_{|mn|}\, \omega^m \wedge \omega^n}$$

He obtains the 2nd structure equation as follows:

Take the exterior derivative of Eq. (35.2) and use Eq. (35.2):

$$d(d\mathbf{e}_i) = d(\mathbf{e}_j \otimes \omega^j{}_i)$$

$$= \mathbf{e}_j\, d\omega^j{}_i + d\mathbf{e}_j \wedge \omega^j{}_i$$

$$\boxed{d^2\mathbf{e}_i =} \boxed{\mathbf{e}_k \otimes (d\omega^k{}_i + \omega^k{}_j \wedge \omega^j{}_i) = \mathbf{e}_k R^k{}_{i|mn|}\, \omega^m \wedge \omega^n}$$

Question: What is the <u>essence</u> of the cause of his successful line of reasoning?

Answer: ...