LECTURE 36

36.1

I Metric Tensor Field

Metric compatibility via

a) covariant derivative

b) differential form

II. Metric compatibility + zero torsion imply

a) a unique parallel transport

b) Christoffel symbols for this parallel transport relative to a coordinate basis.

(36.2)

I. Metric

Parallelism is a condition for the construction of infinitesimal parallelograms from the vectors in the tangent space at each point. However, in spite of the prevalence of this condition, there is no means for

(i) comparing the lengths of the diagonals or of the adjacent sides of any such parallelogram, nor for
(ii) specifying the angle between such sides.

The lack of these is remedied by introducing the concept of a metric

Definition ("Metric")
A <u>metric tensor field</u> is an assignment of

$$g = \text{"}\cdot\text{"} \equiv ds^2 = g_{ij}\,\omega^i \otimes \omega^j \qquad g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$$

to each tangent space of the manifold.
Thus, for $\mathbf{v}, \mathbf{w} \in T_p(M)$ one has

$$\begin{aligned} \mathbf{v}\cdot\mathbf{w} = g(\mathbf{v},\mathbf{w}) &= g_{ij}\,\langle \omega^i | \mathbf{v}\rangle \langle \omega^j | \mathbf{w}\rangle \\ &= g_{ij}\, v^i w^j \end{aligned}$$

in every vector space $T_p(M)$.
From the knowledge of the concept of a metric and that of the torsion of a prevailing parallelism one deduces the following
Proposition ("Unique parallel transport")
A metric tensor field determines a unque torsionless parallelism.
The line of reasoning leading to this conclusion is a 4 step process.

STEP 1

Let $\mathbb{V}$ and $\mathbb{W}$ be two differentiable vector fields. Their inner product $\mathbb{V}\cdot\mathbb{W}=g(\mathbb{V},\mathbb{W})$ in each tangent space is a scalar field on the manifold. In moving from point $\mathcal{P}$ to point $\mathcal{P}+\Delta\mathcal{P}$ the change in this scalar field is

$$\mathbb{V}_{\mathcal{P}+\Delta\mathcal{P}}\cdot\mathbb{W}_{\mathcal{P}+\Delta\mathcal{P}}-\mathbb{V}_{\mathcal{P}}\cdot\mathbb{W}_{\mathcal{P}}\equiv\Delta(\mathbb{V}\cdot\mathbb{W}) \qquad (36.1)$$

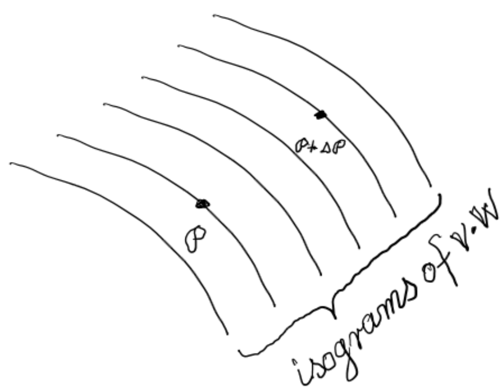

Figure 36.1: Isograms of the scalar field $\mathbb{V}\cdot\mathbb{W}$.

Let the displacement from $\mathcal{P}$ to $\mathcal{P}+\Delta\mathcal{P}$ be generated by the vector $u_{\mathcal{P}}$. Thus

$$\mathbb{V}_{\mathcal{P}+\Delta\mathcal{P}}\cdot\mathbb{W}_{\mathcal{P}+\Delta\mathcal{P}}=e^{\Delta t\, u}\,\mathbb{V}_{\mathcal{P}}\cdot\mathbb{V}_{\mathcal{P}}$$

as was done on page 26.8 of Lecture 26. Apply this Taylor series to Eq. (36.1) on page 36.3 and let $\Delta t \ll 1$. One obtains

$$\Delta(\mathbb{V}\cdot\mathbb{W})=\Delta t\, u(\mathbb{V}\cdot\mathbb{W})\Big|_{\mathcal{P}}+(\text{terms of higher order in } \Delta t).$$

Retain only the principle linear part of this expansion:

$$\Delta(\mathbb{V}\cdot\mathbb{W}) = \Delta t\, \nabla_{u}(\mathbb{V}\cdot\mathbb{W}) \qquad (36.2)$$

This process is depicted in Figure 36.2 below

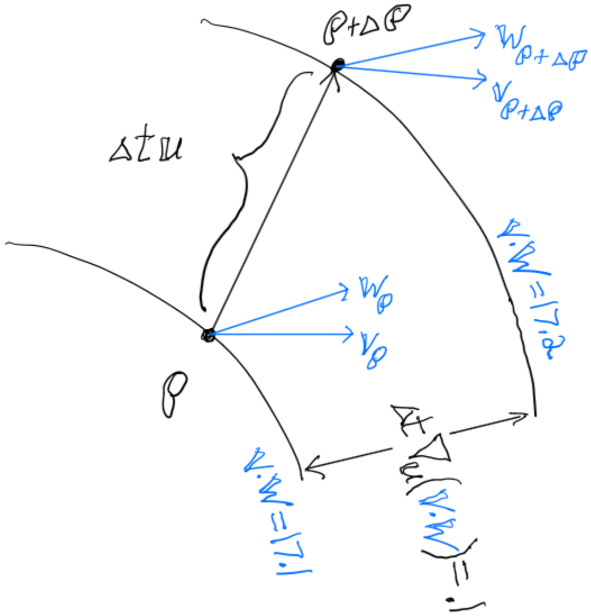

Figure 36.2: Two isograms of the scalar field $V \cdot W$.

STEP 2

The to-be-constructed law of parallelism will be mathematize by two parallel vector fields $Z(t)$ and $X(t)$ along the curve through P and $P+\Delta P$

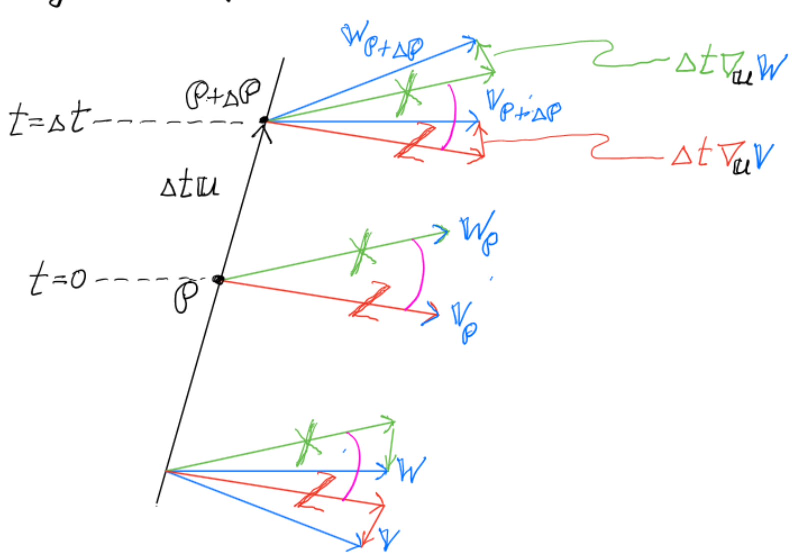

Figure 36.3: Preexisting vector fields $\mathbf{v}$ and $\mathbf{w}$ on the curve whose tangent is $\mathbf{u}(t)$. The to-be-constructed vector fields $\mathbf{X}(t)$ and $\mathbf{Z}(t)$ enforce the prevelance of parallelism by satisfying

$$\nabla_{\mathbf{u}}\mathbf{X} = 0 \qquad \mathbf{X}(t=0) = \mathbf{w}_{\mathcal{P}}$$

and

$$\nabla_{\mathbf{u}}\mathbf{Z} = 0 \qquad \mathbf{Z}_{\mathbf{u}}(t=0) = \mathbf{v}_{\mathcal{P}}$$

along the curve

However, they must also be compatible with the given metric. This condition is expressed by the requirement that their inner product be constant along the curve, i.e.

$$\boxed{\mathbf{X}(t)\cdot\mathbf{Z}(t) = \mathbf{X}(0)\cdot\mathbf{Z}(0) = \mathbf{w}_{\mathcal{P}}\cdot\mathbf{v}_{\mathcal{P}}} \qquad (36.3)$$

for all t. This is the metric compatibility condition

The given preexisting vectors at $\mathcal{P}+\Delta\mathcal{P}$ are $\mathbf{v}_{\mathcal{P}+\Delta\mathcal{P}}$ and $\mathbf{w}_{\mathcal{P}+\Delta\mathcal{P}}$. As depicted in Figure 36.3, they differ from the parallel translates $\mathbf{X}(\Delta t)$ and $\mathbf{Z}(\Delta t)$ by $\Delta t\,\nabla_{\mathbf{u}}\mathbf{w}$ and $\Delta t\,\nabla_{\mathbf{u}}\mathbf{v}$ respectively. In fact one has

$$\mathbf{v}_{\mathcal{P}+\Delta\mathcal{P}} = \mathbf{Z}(\Delta t) + \Delta t\,\nabla_{\mathbf{u}}\mathbf{v}$$

$$\mathbf{w}_{\mathcal{P}+\Delta\mathcal{P}} = \mathbf{X}(\Delta t) + \Delta t\,\nabla_{\mathbf{u}}\mathbf{w}$$

The change in the inner product $\mathbf{v}\cdot\mathbf{w}$ along the curve, Eqs. (36.1)-(36.2) on page 36.3, is therefore

$$\left(\mathbf{Z}(\Delta t) + \Delta t\,\nabla_{\mathbf{u}}\mathbf{v}\right)\cdot\left(\mathbf{X}(\Delta t) + \Delta t\,\nabla_{\mathbf{u}}\mathbf{w}\right) - \mathbf{v}_{\mathcal{P}}\cdot\mathbf{w}_{\mathcal{P}} = \Delta t\,\nabla_{\mathbf{u}}(\mathbf{v}\cdot\mathbf{w})$$

In light of Eq. (36.3) on page 36.5, namely, $\mathbf{v}_{\mathcal{P}}\cdot\mathbf{w}_{\mathcal{P}} = \mathbf{X}(\Delta t)\cdot\mathbf{Z}(\Delta t)$, one finds

$$\Delta t\,\nabla_{\mathbf{u}}\mathbf{v}\cdot\mathbf{X}(\Delta t) + \mathbf{Z}(\Delta t)\cdot\Delta t\,\nabla_{\mathbf{u}}\mathbf{w} = \Delta t\,\nabla_{\mathbf{u}}(\mathbf{v}\cdot\mathbf{w}) \qquad (36.4)$$

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From Figure 36.3 one infers that

$$X(\Delta t) = W_{\mathcal{P}+\Delta\mathcal{P}} - \Delta t\, \nabla_u W$$
$$Z(\Delta t) = V_{\mathcal{P}+\Delta\mathcal{P}} - \Delta t\, \nabla_u V$$

Introduce these vectors into Eq.(36.4), neglect higher order terms and find that

$$\boxed{\nabla_u V \cdot W + V \cdot \nabla_u W = \nabla_u (V \cdot W)}$$ ("metric compatibility condition")

This is the condition for parallelism, as expressed by its covariant derivative ∇, to be compatible with the metric. This equation holds for all vectors u, V, and W. As illustrates in Figure 36.3, it mathematizes the fact that lengths and angles are preserved under parallel transport. Such a law preserves the metric structure between $T_{\mathcal{P}}(M)$ and $T_{\mathcal{P}+\Delta\mathcal{P}}(M)$ under parallel transport.

STEP 3

The restriction of metric compatibility on any law of parallel transport is brought to the forefront by applying this compatibility to the basis vectors. Setting

$$u = e_k$$
$$V = e_i$$
$$W = e_j$$

one finds that

$$(\nabla_k e_i) \cdot e_j + e_i \cdot (\nabla_k e_j) = \nabla_k (e_i \cdot e_j)$$

In terms of the "Christoffel symbols of the 2nd kind" this becomes

$$e_\ell \Gamma^\ell{}_{ik} \cdot e_j + e_i \cdot e_\ell \Gamma^\ell{}_{jk} = \nabla_k (g_{ij}) \qquad (36.4)$$

For the purpose of unit-economy introduce

$$\Gamma^{\ell}{}_{ik}\, g_{\ell j} \equiv \Gamma_{j\,ik}$$
$$\Gamma^{\ell}{}_{jk}\, g_{i\ell} \equiv \Gamma_{i\,jk}$$

In terms of the "Christoffel symbols of the 1st kind" metric compatibility is

$$\Gamma_{j\,ik} + \Gamma_{i\,jk} = \nabla_k (g_{ij})$$

STEP 4

In order to obtain the unique torsionless parallel transport apply the zero torsion condition

$$0 = \mathbb{T}(e_i, e_j) = \nabla_{e_i} e_j - \nabla_{e_j} e_i - [e_i, e_j]$$
$$= e_\ell \left(\Gamma^{\ell}{}_{ji} - \Gamma^{\ell}{}_{ij}\right) - e_\ell\, c^{\ell}{}_{ij}$$

For the sake of mathematical simplicity focus on a coordinate basis whose commutator always vanishes:

$$[e_i, e_j] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

Consequently,

$$\Gamma^{\ell}{}_{ji} = \Gamma^{\ell}{}_{ij}.$$

In other words, for zero torsion, relative to a coordinate basis, the Christoffel symbols are symmetric:

$$\Gamma^{\ell}{}_{(ji)} = \Gamma^{\ell}{}_{(ij)}.$$

It follows that torsionless metric compatibility – relative to a coordinate bases – implies

$$\left.\begin{aligned} \cancel{\Gamma_{j(ik)}} + \Gamma_{i(jk)} &= \partial_k(g_{ij}) \\ \Gamma_{i(kj)} + \cancel{\Gamma_{k(ij)}} &= \partial_j(g_{ik}) \\ -\cancel{\Gamma_{k(ji)}} - \cancel{\Gamma_{j(ki)}} &= -\partial_i(g_{jk}) \end{aligned}\right\} \text{cyclic permutation} \qquad (36.8)$$

Thus,

$$\Gamma_{i(jk)} = \frac{1}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right)$$

or

$$\boxed{\Gamma^{\ell}{}_{jk} = \frac{g^{\ell i}}{2}\left(g_{ij,k} + g_{ik,j} - g_{jk,i}\right)}$$

These are the "Christoffel symbols of the 2nd kind". They make up the law

$$d\mathbf{e}_j = \mathbf{e}_\ell \, \Gamma^{\ell}{}_{jk} \, dx^k$$

where

$$\Gamma^{\ell}{}_{jk} = \Gamma^{\ell}{}_{kj}$$

whenever torsion = zero as determined uniquely by the metric.

Summary

Regardless of whether the torsion tensor of parallelism vanishes or not, parallelism's metric compatibility is mathematized by the requirement that

$$\boxed{\nabla_u(\mathbb{V} \cdot \mathbb{W}) = \nabla_u \mathbb{V} \cdot \mathbb{W} + \mathbb{V} \cdot \nabla_u \mathbb{W}}$$

Once one has chosen a basis, say $\{\mathbf{e}_k\}$, this requirement assumes the form

$$\nabla_{\mathbf{e}_k}(g_{ij}) = g_{i\ell}\,\Gamma^{\ell}{}_{jk} + g_{\ell j}\,\Gamma^{\ell}{}_{ik}.$$

These are precisely the components of the covariant derivative of the metric tensor $g = g_{ij}\,\omega^i \otimes \omega^j$;

$$\nabla_{e_k} g = \nabla_{e_k}(g_{ij}) + g_{ij}(\nabla_{e_k}\omega^i)\otimes\omega^j + g_{ij}\,\omega^i\otimes\nabla_{e_k}\omega^j. \quad (36.5) \quad (\text{"product rule"})$$

In light of the covariant derivative of the dual basis elements, Eq. (31.2) on page 31.5, the metric compatibility requirement is mathematized by the statement

$$\boxed{\nabla_{e_k} g = 0}:$$

The <u>metric tensor is always covariantly constant.</u>

If the chosen basis is a coordinate basis, $\{e_k = \frac{\partial}{\partial x^k}\}$, this requirement is

$$\frac{\partial g_{ij}}{\partial x^k} - g_{i\ell}\Gamma^{\ell}_{jk} - g_{j\ell}\Gamma^{\ell}_{ik} = 0$$

or, in light of Eq. (31.3) on page 31.5,

$$\boxed{g_{ij;k} = 0}$$

even for non-zero torsion (i.e $\Gamma^{\ell}_{jk} - \Gamma^{\ell}_{kj} \neq 0$). These are the components of the covariant derivative, Eq. (36.5) on page 36.9.