37.1

LECTURE 37

Four fundamental equations of differential geometry:

I. Via differential operators

II. Via differential forms.

In MTW read §14.5 and §13.3

The 4 Fundamental Equations of Differential Geometry

The first two are based on the phenomenon of parallel transport apart from any reference to lengths and angles. The second two are also based on parallel transport, but they introduce new concepts: length and angle. Their introduction narrows the scope of the types of parallelisms.

Requiring the invariance of the squared lengths of vectors, and hence of their inner products, under parallel displacement restricts the parallelisms to those which are metric compatible.

Applying this requirement to the perimeter of an area permeated by non-zero curvature results in a non-zero rotation, the same for every vector circum-navigating that perimeter.

The four fundamental equations of differential geometry were first stated by Elie Cartan in terms of differential forms, and subsequently after WWII by a number of people in terms of the covariant derivative.

I. In terms of the covariant derivative the 4 fundamental equations are:

① Torsion: $\nabla_{\mathbb{U}}\mathbb{V} - \nabla_{\mathbb{V}}\mathbb{U} - [\mathbb{U},\mathbb{V}] = \mathbb{T}(\mathbb{U},\mathbb{V}) = \mathbb{e}_i T^i{}_{mn} u^m v^n$

\footnote{Being a vector, $\mathbb{e}_i T^i{}_{mn} u^m v^n$ is a displacement generator. See Lecture 26.}

② Curvature: $\{\nabla_{\mathbb{U}}\nabla_{\mathbb{V}} - \nabla_{\mathbb{V}}\nabla_{\mathbb{U}} - \nabla_{[\mathbb{U},\mathbb{V}]}\}\mathbb{W} \equiv \mathcal{R}(\mathbb{U},\mathbb{V})\mathbb{W} = \mathbb{R}(\cdots,\mathbb{W},\mathbb{U},\mathbb{V}) = \mathbb{e}_i \underbrace{R^i{}_{jmn} u^m v^n} w^j$

\footnote{Being a rank $\binom{1}{1}$ tensor, $\mathbb{e}_i R^i{}_{jmn} u^m v^n \underset{\sim}{\omega}^j$ is a generalized rotation generator. Upon operating on $\mathbb{W}$ it generates an infinitesimal rotational change in $\mathbb{W}$.}

③ Metric compatibility: $\nabla_{\mathbb{U}}(\mathbb{V}\cdot\mathbb{W}) = (\nabla_{\mathbb{U}}\mathbb{V})\cdot\mathbb{W} + \mathbb{V}\cdot(\nabla_{\mathbb{U}}\mathbb{W})$

\footnote{ When operating on inner products, $\nabla_{\mathbb{u}}$ obeys the "Leibnitz" (product) rule.} (37.3)

④ Metric-induced anti-symmetry of Riemann curvature

$$0 = \mathcal{R}(\mathbb{u},\mathbb{v})(\mathbb{w}\cdot\mathbb{x}) = [\mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{w}]\cdot\mathbb{x} + \mathbb{w}\cdot[\mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{x}] \qquad (37.1)$$

\footnote{ When operating on inner products, $\mathcal{R}(\mathbb{u},\mathbb{v})$ obeys the "Leibnitz" (product) rule.}

$$= [R_{\ell jmn} w^j u^m v^n] x^\ell + w^\ell [R_{\ell jmn} x^j u^m v^n] = (R_{\ell jmn} + R_{j\ell mn}) x^\ell w^j u^m v^n$$

$$\Longrightarrow \quad 0 = R_{\ell jmn} + R_{j\ell mn} \quad \Longrightarrow \quad R_{\ell jmn} = -R_{j\ell mn}$$

Remark 1

The l.h.s. of Eq. (37.1) is zero because $\mathbb{w}\cdot\mathbb{x} \equiv f$ is a scalar:

$$\begin{aligned}\mathcal{R}(\mathbb{u},\mathbb{v})(\mathbb{w}\cdot\mathbb{x}) &= \{\nabla_{\mathbb{u}}\nabla_{\mathbb{v}} - \nabla_{\mathbb{v}}\nabla_{\mathbb{u}} - \nabla_{[\mathbb{u},\mathbb{v}]}\} f \\ &= \partial_{\mathbb{u}}\partial_{\mathbb{v}} f - \partial_{\mathbb{v}}\partial_{\mathbb{u}} f - (\mathbb{u}\mathbb{v} - \mathbb{v}\mathbb{u}) f \\ &= \partial_{\mathbb{u}}\partial_{\mathbb{v}} f - \partial_{\mathbb{v}}\partial_{\mathbb{u}} f - (\partial_{\mathbb{u}}\partial_{\mathbb{v}} - \partial_{\mathbb{v}}\partial_{\mathbb{u}}) f \\ &= 0\end{aligned}$$

The r.h.s. of Eq(37.1) is

$$\mathcal{R}(\mathbb{u},\mathbb{v})(\mathbb{w}\cdot\mathbb{x}) = \{\underset{①}{\nabla_{\mathbb{u}}\nabla_{\mathbb{v}}} - \underset{②}{\nabla_{\mathbb{v}}\nabla_{\mathbb{u}}} - \underset{③}{\nabla_{[\mathbb{u},\mathbb{v}]}}\}(\mathbb{w}\cdot\mathbb{x})$$

Apply metric compatibility to ① and ② twice, and to ③ once.
The result after cancellation is

$$0 = \mathcal{R}(\mathbb{u},\mathbb{v})(\mathbb{w}\cdot\mathbb{x}) = \underbrace{[\{\nabla_{\mathbb{u}}\nabla_{\mathbb{v}} - \nabla_{\mathbb{v}}\nabla_{\mathbb{u}} - \nabla_{[\mathbb{u},\mathbb{v}]}\}\mathbb{w}]}_{\mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{w}}\cdot\mathbb{x} + \mathbb{w}\cdot\underbrace{[\{\nabla_{\mathbb{u}}\nabla_{\mathbb{v}} - \nabla_{\mathbb{v}}\nabla_{\mathbb{u}} - \nabla_{[\mathbb{u},\mathbb{v}]}\}\mathbb{x}]}_{\mathcal{R}(\mathbb{u},\mathbb{v})\mathbb{x}}$$

This is the 4th fundamental equation for the metric-induced antisymmetry of the curvature.

Remark 2

Given a metric tensor field $g_{ij}\,\omega^i\otimes\omega^j$, the calculation of the curvature consists calculating its components relative to the given (or chosen) basis $\{e_k\}$. (37.4)

This calculation consist of setting

$$u = e_i \,,\ v = e_j \,,\ w = e_k \qquad \left.\begin{matrix} i \\ j \\ k \end{matrix}\right\} = 0,1,2,3$$

and then calculating

$$\Gamma^i{}_{jk} = \tfrac{1}{2}\, g^{il}\left(g_{lj,k} + g_{lk,j} - g_{jk,l}\right)$$

and its derivative $\Gamma^i{}_{jk,l}$ in order to obtain the components of the 1st and 2nd covariant derivatives that go into the curvature calculation. This is a formidable task even in the presence of spherical symmetry, for example. This is because for such configurations many of the requisite calculations for the curvature give uninteresting results, namely zero.

II. Cartan avoided the calculation of uninteresting results by stating the four fundamental equations in terms of differential forms. Starting with his unit tensor and the law of parallel transport,

$$dP = e_i\,\omega^i \quad \text{and} \quad de_i = e_j\,\omega^j{}_i \,,$$

Cartan expressed all 4 fundamental equation in terms of differential forms.

① Torsion 2-form: Cartan takes the exterior derivative of dP:

$$d(dP) = e_i\left(d\omega^i + \omega^i{}_j \wedge \omega^j\right) \equiv e_i \otimes \underset{\sim}{\Omega}^i = e_i\, T^i{}_{|mn|}\, \omega^m \wedge \omega^n$$

② Curvature 2-form: take the exterior derivative of de_i:

$$d(de_i) = e_j\left(d\omega^j{}_i + \omega^j{}_\ell \wedge \omega^\ell{}_i\right) \equiv e_j \Omega^j{}_i = e_j R^j{}_{i|mn|}\, \omega^m \wedge \omega^n \qquad (37.5)$$

③ Metric compatibility

$$d g_{ij} = d(e_i \cdot e_j) = (de_i)\cdot e_j + e_i \cdot (de_j)$$
$$= e_\ell\, \omega^\ell{}_i \cdot e_j + e_i \cdot e_\ell\, \omega^\ell{}_j$$
$$= g_{j\ell}\, \omega^\ell{}_i + g_{i\ell}\, \omega^\ell{}_j$$

$$\boxed{d g_{ij} = g_{i\ell}\omega^\ell{}_j + g_{j\ell}\omega^\ell{}_i} \qquad (37.2)$$

④ Curvature anti symmetry

$$0 = g_{i\ell}\, \Omega^\ell{}_j + g_{j\ell}\, \Omega^\ell{}_i$$
$$0 = \Omega_{ij} + \Omega_{ji}$$

Remark 3

Exterior derivative of metric compatibility leads to curvature anti-symmetry

$$0 = d(d g_{ij}) = d\left(g_{i\ell}\omega^\ell{}_j + g_{j\ell}\,\omega^\ell{}_i\right)$$
$$= d g_{i\ell} \wedge \omega^\ell{}_j + g_{i\ell}\, d\omega^\ell{}_j + d g_{j\ell} \wedge \omega^\ell{}_i + g_{j\ell}\, d\omega^\ell{}_i$$
$$= \left(g_{ik}\omega^k{}_\ell + \cancel{g_{\ell k}\omega^k{}_i}\right) \wedge \omega^\ell{}_j + \left(g_{jk}\,\omega^k{}_\ell + \cancel{g_{\ell k}}\,\omega^k{}_j\right) \wedge \omega^\ell{}_i$$
$$g_{i\ell}\, d\omega^\ell{}_j \qquad\qquad + g_{j\ell}\, d\omega^\ell{}_i$$

$(k \leftrightarrow \ell)$

$$= g_{i\ell}\left(d\omega^\ell{}_j + \omega^\ell{}_k \wedge \omega^k{}_j\right) + g_{j\ell}\left(d\omega^\ell{}_i + \omega^\ell{}_k \wedge \omega^k{}_i\right)$$
$$0 = g_{i\ell}\, \Omega^\ell{}_j + g_{j\ell}\, \Omega^\ell{}_i$$

$$\boxed{0 = \Omega_{ij} + \Omega_{ji}}$$

Summary

(37.6)

In calculating the curvature it is difficult to point to a method more efficient than the one based on utilizing the 4 fundamental equation with in the framework of exterior calculus of diffential forms. The method is a four-step process.

① Given a specific metric $g = g_{ij}\,\omega^i \otimes \omega^j$, choose an appropriate basis $\{\omega^i\}$ relative to which represent g.

② Calculate the exterior derivatives $d\omega^i$ and determine algebraically the connection 1-forms $\underset{\sim}{\omega}^i{}_j$ from the zero torsion condition $d\omega^i + \omega^i{}_j = 0$ and the metric compatibility condition.

$$\omega^i{}_j \wedge \omega^j = -\,d\omega^i$$

$$g_{il}\,\omega^l{}_j + g_{jl}\,\omega^l{}_i = d\,g_{ij}$$

This is a problem in linear algebra whose simplicity depends on one's judicious choice of the metric coefficients g_{ij}, which in turn depend on one's choice of the basis $\{\omega^i\}$ in step ①.

③ Calculate the curvature 2-forms

$$\underset{\sim}{d\omega}^i{}_j + \underset{\sim}{\omega}^i{}_l \wedge \underset{\sim}{\omega}^l{}_j \equiv \underset{\sim}{\Omega}^i{}_j.$$

④ Read out the components $R^i{}_{jmn}$ of the curvature tensor

$$\Omega^i{}_j = R^i{}_{j|mn|}\,\omega^m \wedge \omega^n$$

COMMENT 1

Being based on the metric compatibility, the curvature is anti-symmetric in its first two coordinate component indeces:

$$R_{ijmn} = -R_{jimn}$$

COMMENT 2

The curvature 2-forms $\Omega^i{}_j$ contain only those terms which have non-zero curvature components. All others are zero by virtue of the linear independence of the tensor basis elements $\{\omega^m \wedge \omega^n\}$.