VALUES OF T_μ^ν AND THE CHRISTOFFEL SYMBOLS FOR A LINE ELEMENT OF CONSIDERABLE GENERALITY

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Communicated March 20, 1933

In the general theory of relativity the mechanical properties of any region of the universe are expressed by the energy-momentum tensor, T_μ^ν, which is itself calculable from the form of the line element, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, applicable to that region. The expressions for T_μ^ν in the most general case, in which all the $g_{\mu\nu}$ are arbitrary functions of the four coördinates, x^1, x^2, x^3, x^4, are exceedingly complicated, but considerable simplification is introduced if it is assumed that $g_{\mu\nu}(\mu \neq \nu) = 0$. The resulting line element still possesses a large amount of generality, and in the applications of the theory particular forms of it have, in fact, usually been employed. It therefore seems desirable to publish the general expressions for the energy-momentum tensor corresponding to this line element, and it is the purpose of this paper to give them, together with the associated values of the Christoffel symbols of the second kind, in the form best suited for application. The calculations, which are somewhat long, have kindly been checked by Mr. C. C. Steffens of the California Institute of Technology, and the proofs have been carefully read, so that the results may be used with considerable confidence. It is hoped that their publication will save labor for those working in this field.

The expression for the line element is taken as

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2,$$

where A, B, C and D are any functions of x^1, x^2, x^3 and x^4. Mathematically these functions may be positive or negative, real or imaginary,* but in ordinary applications, in which x^4 is the time-like coördinate, they will clearly always be positive and real. The non-vanishing components of the metric tensor and its contravariant associate are obviously as follows:

$$g_{11} = -A;\ g_{22} = -B;\ g_{33} = -C;\ g_{44} = +D$$

$$g^{11} = -\frac{1}{A};\ g^{22} = -\frac{1}{B};\ g^{33} = -\frac{1}{C};\ g^{44} = +\frac{1}{D};$$

and the determinant, g, is $-ABCD$.

Christoffel Symbols.—These are defined by the expression

$$\{\mu\nu, \sigma\} = \tfrac{1}{2}\, g^{\sigma\lambda}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\nu} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\lambda}\right).$$

* It is assumed that they possess first and second differential coefficients with respect to each of the coördinates.

Their values are

$\{11,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^1}$	$\{21,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^2}$	$\{31,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^3}$	$\{41,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^4}$
$\{11,2\} = -\frac{1}{2B}\frac{\partial A}{\partial x^2}$	$\{21,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^1}$	$\{31,2\} = 0$	$\{41,2\} = 0$
$\{11,3\} = -\frac{1}{2C}\frac{\partial A}{\partial x^3}$	$\{21,3\} = 0$	$\{31,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^1}$	$\{41,3\} = 0$
$\{11,4\} = +\frac{1}{2D}\frac{\partial A}{\partial x^4}$	$\{21,4\} = 0$	$\{31,4\} = 0$	$\{41,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^1}$
$\{12,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^2}$	$\{22,1\} = -\frac{1}{2A}\frac{\partial B}{\partial x^1}$	$\{32,1\} = 0$	$\{42,1\} = 0$
$\{12,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^1}$	$\{22,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^2}$	$\{32,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^3}$	$\{42,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^4}$
$\{12,3\} = 0$	$\{22,3\} = -\frac{1}{2C}\frac{\partial B}{\partial x^3}$	$\{32,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^2}$	$\{42,3\} = 0$
$\{12,4\} = 0$	$\{22,4\} = +\frac{1}{2D}\frac{\partial B}{\partial x^4}$	$\{32,4\} = 0$	$\{42,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^2}$
$\{13,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^3}$	$\{23,1\} = 0$	$\{33,1\} = -\frac{1}{2A}\frac{\partial C}{\partial x^1}$	$\{43,1\} = 0$
$\{13,2\} = 0$	$\{23,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^3}$	$\{33,2\} = -\frac{1}{2B}\frac{\partial C}{\partial x^2}$	$\{43,2\} = 0$
$\{13,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^1}$	$\{23,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^2}$	$\{33,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^3}$	$\{43,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^4}$
$\{13,4\} = 0$	$\{23,4\} = 0$	$\{33,4\} = +\frac{1}{2D}\frac{\partial C}{\partial x^4}$	$\{43,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^3}$
$\{14,1\} = +\frac{1}{2A}\frac{\partial A}{\partial x^4}$	$\{24,1\} = 0$	$\{34,1\} = 0$	$\{44,1\} = +\frac{1}{2A}\frac{\partial D}{\partial x^1}$
$\{14,2\} = 0$	$\{24,2\} = +\frac{1}{2B}\frac{\partial B}{\partial x^4}$	$\{34,2\} = 0$	$\{44,2\} = +\frac{1}{2B}\frac{\partial D}{\partial x^2}$
$\{14,3\} = 0$	$\{24,3\} = 0$	$\{34,3\} = +\frac{1}{2C}\frac{\partial C}{\partial x^4}$	$\{44,3\} = +\frac{1}{2C}\frac{\partial D}{\partial x^3}$

$$\{14,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^1}\ \Bigg|\ \{24,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^2}\ \Bigg|\ \{34,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^3}\ \Bigg|\ \{44,4\} = +\frac{1}{2D}\frac{\partial D}{\partial x^4}$$

Energy-Momentum Tensor, T^ν_μ.—This tensor is defined by the expression

$$-8\pi T^\nu_\mu = G^\nu_\mu - \tfrac{1}{2}g^\nu_\mu G + g^\nu_\mu \lambda$$

where G^ν_μ is the contracted Riemann-Christoffel tensor, G is the invariant,

$g^{\mu\nu}G_{\mu\nu}$, and λ is the cosmological constant. The values of $-8\pi T^\nu_\mu$ are as follows:

$$\begin{aligned}-8\pi T^1_1 = \frac{1}{2}\Bigg[&\frac{1}{BC}\left(\frac{\partial^2 B}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^2)^2}\right)-\frac{1}{BD}\left(\frac{\partial^2 B}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^2)^2}\right)\\&-\frac{1}{CD}\left(\frac{\partial^2 C}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^3)^2}\right)\Bigg]\\
-\frac{1}{4}\Bigg[&\frac{1}{BC^2}\left\{\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^2}\right)^2\right\}+\frac{1}{CB^2}\left\{\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^2}+\left(\frac{\partial B}{\partial x^3}\right)^2\right\}\\
&-\frac{1}{BD^2}\left\{\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^2}\right)^2\right\}+\frac{1}{DB^2}\left\{\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^2}-\left(\frac{\partial B}{\partial x^4}\right)^2\right\}\\
&-\frac{1}{CD^2}\left\{\frac{\partial C}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^3}\right)^2\right\}+\frac{1}{DC^2}\left\{\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^3}-\left(\frac{\partial C}{\partial x^4}\right)^2\right\}\\
&-\frac{1}{BCD}\left\{\frac{\partial C}{\partial x^2}\frac{\partial D}{\partial x^2}+\frac{\partial B}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{\partial B}{\partial x^4}\frac{\partial C}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1}\\
&-\frac{1}{ABD}\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^1}-\frac{1}{ACD}\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^1}\Bigg]+\end{aligned}$$

$$\begin{aligned}-8\pi T^2_2 = \frac{1}{2}\Bigg[&\frac{1}{AC}\left(\frac{\partial^2 A}{\partial(x^3)^2}+\frac{\partial^2 C}{\partial(x^1)^2}\right)-\frac{1}{AD}\left(\frac{\partial^2 A}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^1)^2}\right)\\&-\frac{1}{CD}\left(\frac{\partial^2 C}{\partial(x^4)^2}-\frac{\partial^2 D}{\partial(x^3)^2}\right)\Bigg]\\
-\frac{1}{4}\Bigg[&\frac{1}{AC^2}\left\{\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^3}+\left(\frac{\partial C}{\partial x^1}\right)^2\right\}+\frac{1}{CA^2}\left\{\frac{\partial C}{\partial x^1}\frac{\partial A}{\partial x^1}+\left(\frac{\partial A}{\partial x^3}\right)^2\right\}\\
&-\frac{1}{AD^2}\left\{\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^1}\right)^2\right\}+\frac{1}{DA^2}\left\{\frac{\partial D}{\partial x^1}\frac{\partial A}{\partial x^1}-\left(\frac{\partial A}{\partial x^4}\right)^2\right\}\\
&-\frac{1}{CD^2}\left\{\frac{\partial C}{\partial x^4}\frac{\partial D}{\partial x^4}-\left(\frac{\partial D}{\partial x^3}\right)^2\right\}+\frac{1}{DC^2}\left\{\frac{\partial D}{\partial x^3}\frac{\partial C}{\partial x^3}-\left(\frac{\partial C}{\partial x^4}\right)^2\right\}\\
&-\frac{1}{ACD}\left\{\frac{\partial C}{\partial x^1}\frac{\partial D}{\partial x^1}+\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^3}-\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^4}\right\}-\frac{1}{ABC}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2}\\
&-\frac{1}{ABD}\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^2}-\frac{1}{BCD}\frac{\partial C}{\partial x^2}\frac{\partial D}{\partial x^2}\Bigg]+\lambda\end{aligned}$$

$$-8\pi T_3^3 = \frac{1}{2}\left[\frac{1}{AB}\left(\frac{\partial^2 A}{\partial(x^2)^2} + \frac{\partial^2 B}{\partial(x^1)^2}\right) - \frac{1}{AD}\left(\frac{\partial^2 A}{\partial(x^4)^2} - \frac{\partial^2 D}{\partial(x^1)^2}\right) - \frac{1}{BD}\left(\frac{\partial^2 B}{\partial(x^4)^2} - \frac{\partial^2 D}{\partial(x^2)^2}\right)\right]$$

$$-\frac{1}{4}\left[\frac{1}{AB^2}\left\{\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2} + \left(\frac{\partial B}{\partial x^1}\right)^2\right\} + \frac{1}{BA^2}\left\{\frac{\partial B}{\partial x^1}\frac{\partial A}{\partial x^1} + \left(\frac{\partial A}{\partial x^2}\right)^2\right\}\right.$$

$$-\frac{1}{AD^2}\left\{\frac{\partial A}{\partial x^4}\frac{\partial D}{\partial x^4} - \left(\frac{\partial D}{\partial x^1}\right)^2\right\} + \frac{1}{DA^2}\left\{\frac{\partial D}{\partial x^1}\frac{\partial A}{\partial x^1} - \left(\frac{\partial A}{\partial x^4}\right)^2\right\}$$

$$-\frac{1}{BD^2}\left\{\frac{\partial B}{\partial x^4}\frac{\partial D}{\partial x^4} - \left(\frac{\partial D}{\partial x^2}\right)^2\right\} + \frac{1}{DB^2}\left\{\frac{\partial D}{\partial x^2}\frac{\partial B}{\partial x^2} - \left(\frac{\partial B}{\partial x^4}\right)^2\right\}$$

$$-\frac{1}{ABD}\left\{\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^1} + \frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^2} - \frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^4}\right\} - \frac{1}{ABC}\frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}$$

$$\left. - \frac{1}{ACD}\frac{\partial A}{\partial x^3}\frac{\partial D}{\partial x^3} - \frac{1}{BCD}\frac{\partial B}{\partial x^3}\frac{\partial D}{\partial x^3}\right] + \lambda$$

$$-8\pi T_4^4 = \frac{1}{2}\left[\frac{1}{AB}\left(\frac{\partial^2 A}{\partial(x^2)^2} + \frac{\partial^2 B}{\partial(x^1)^2}\right) + \frac{1}{AC}\left(\frac{\partial^2 A}{\partial(x^3)^2} + \frac{\partial^2 C}{\partial(x^1)^2}\right) + \frac{1}{BC}\left(\frac{\partial^2 B}{\partial(x^3)^2} + \frac{\partial^2 C}{\partial(x^2)^2}\right)\right]$$

$$-\frac{1}{4}\left[\frac{1}{AB^2}\left\{\frac{\partial A}{\partial x^2}\frac{\partial B}{\partial x^2} + \left(\frac{\partial B}{\partial x^1}\right)^2\right\} + \frac{1}{BA^2}\left\{\frac{\partial B}{\partial x^1}\frac{\partial A}{\partial x^1} + \left(\frac{\partial A}{\partial x^2}\right)^2\right\}\right.$$

$$+\frac{1}{AC^2}\left\{\frac{\partial A}{\partial x^3}\frac{\partial C}{\partial x^3} + \left(\frac{\partial C}{\partial x^1}\right)^2\right\} + \frac{1}{CA^2}\left\{\frac{\partial C}{\partial x^1}\frac{\partial A}{\partial x^1} + \left(\frac{\partial A}{\partial x^3}\right)^2\right\}$$

$$+\frac{1}{BC^2}\left\{\frac{\partial B}{\partial x^3}\frac{\partial C}{\partial x^3} + \left(\frac{\partial C}{\partial x^2}\right)^2\right\} + \frac{1}{CB^2}\left\{\frac{\partial C}{\partial x^2}\frac{\partial B}{\partial x^2} + \left(\frac{\partial B}{\partial x^3}\right)^2\right\}$$

$$-\frac{1}{ABC}\left\{\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^1} + \frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^2} + \frac{\partial A}{\partial x^3}\frac{\partial B}{\partial x^3}\right\} + \frac{1}{ABD}\frac{\partial A}{\partial x^4}\frac{\partial B}{\partial x^4}$$

$$\left. + \frac{1}{ACD}\frac{\partial A}{\partial x^4}\frac{\partial C}{\partial x^4} + \frac{1}{BCD}\frac{\partial B}{\partial x^4}\frac{\partial C}{\partial x^4}\right] + \lambda$$

$$-8\pi A T_2^1 = -8\pi B T_1^2 =$$

$$-\frac{1}{2}\left[\frac{1}{C}\frac{\partial^2 C}{\partial x^1 \partial x^2} + \frac{1}{D}\frac{\partial^2 D}{\partial x^1 \partial x^2}\right]$$

$$+\frac{1}{4}\left[\frac{1}{C^2}\frac{\partial C}{\partial x^1}\frac{\partial C}{\partial x^2} + \frac{1}{D^2}\frac{\partial D}{\partial x^1}\frac{\partial D}{\partial x^2} + \frac{1}{AC}\frac{\partial A}{\partial x^2}\frac{\partial C}{\partial x^1} + \frac{1}{AD}\frac{\partial A}{\partial x^2}\frac{\partial D}{\partial x^1}\right.$$

$$\left. + \frac{1}{BC}\frac{\partial B}{\partial x^1}\frac{\partial C}{\partial x^2} + \frac{1}{BD}\frac{\partial B}{\partial x^1}\frac{\partial D}{\partial x^2}\right]$$

$$-8\pi A T_3^1 = -8\pi C T_1^3 =$$

$$
-\frac{1}{2}\left[\frac{1}{B}\,\frac{\partial^2 B}{\partial x^1 \partial x^3}+\frac{1}{D}\,\frac{\partial^2 D}{\partial x^1 \partial x^3}\right]
+\frac{1}{4}\left[\frac{1}{B^2}\,\frac{\partial B}{\partial x^1}\,\frac{\partial B}{\partial x^3}+\frac{1}{D^2}\,\frac{\partial D}{\partial x^1}\,\frac{\partial D}{\partial x^3}+\frac{1}{AB}\,\frac{\partial A}{\partial x^3}\,\frac{\partial B}{\partial x^1}+\frac{1}{AD}\,\frac{\partial A}{\partial x^3}\,\frac{\partial D}{\partial x^1}
+\frac{1}{CB}\,\frac{\partial C}{\partial x^1}\,\frac{\partial B}{\partial x^3}+\frac{1}{CD}\,\frac{\partial C}{\partial x^1}\,\frac{\partial D}{\partial x^3}\right].
$$

$$
-8\pi B T^2_3 = -8\pi C T^3_2 =
-\frac{1}{2}\left[\frac{1}{A}\,\frac{\partial^2 A}{\partial x^2 \partial x^3}+\frac{1}{D}\,\frac{\partial^2 D}{\partial x^2 \partial x^3}\right]
+\frac{1}{4}\left[\frac{1}{A^2}\,\frac{\partial A}{\partial x^2}\,\frac{\partial A}{\partial x^3}+\frac{1}{D^2}\,\frac{\partial D}{\partial x^2}\,\frac{\partial D}{\partial x^3}+\frac{1}{AB}\,\frac{\partial A}{\partial x^2}\,\frac{\partial B}{\partial x^3}+\frac{1}{AC}\,\frac{\partial A}{\partial x^3}\,\frac{\partial C}{\partial x^2}
+\frac{1}{DB}\,\frac{\partial D}{\partial x^2}\,\frac{\partial B}{\partial x^3}+\frac{1}{DC}\,\frac{\partial D}{\partial x^3}\,\frac{\partial C}{\partial x^2}\right]
$$

$$
-8\pi A T^1_4 = +8\pi D T^4_1 =
-\frac{1}{2}\left[\frac{1}{B}\,\frac{\partial^2 B}{\partial x^1 \partial x^4}+\frac{1}{C}\,\frac{\partial^2 C}{\partial x^1 \partial x^4}\right]
+\frac{1}{4}\left[\frac{1}{B^2}\,\frac{\partial B}{\partial x^1}\,\frac{\partial B}{\partial x^4}+\frac{1}{C^2}\,\frac{\partial C}{\partial x^1}\,\frac{\partial C}{\partial x^4}+\frac{1}{AB}\,\frac{\partial A}{\partial x^4}\,\frac{\partial B}{\partial x^1}+\frac{1}{AC}\,\frac{\partial A}{\partial x^4}\,\frac{\partial C}{\partial x^1}
+\frac{1}{DB}\,\frac{\partial D}{\partial x^1}\,\frac{\partial B}{\partial x^4}+\frac{1}{DC}\,\frac{\partial D}{\partial x^1}\,\frac{\partial C}{\partial x^4}\right]
$$

$$
-8\pi B T^2_4 = +8\pi D T^4_2 =
-\frac{1}{2}\left[\frac{1}{A}\,\frac{\partial^2 A}{\partial x^2 \partial x^4}+\frac{1}{C}\,\frac{\partial^2 C}{\partial x^2 \partial x^4}\right]
+\frac{1}{4}\left[\frac{1}{A^2}\,\frac{\partial A}{\partial x^2}\,\frac{\partial A}{\partial x^4}+\frac{1}{C^2}\,\frac{\partial C}{\partial x^2}\,\frac{\partial C}{\partial x^4}+\frac{1}{AB}\,\frac{\partial A}{\partial x^2}\,\frac{\partial B}{\partial x^4}+\frac{1}{AD}\,\frac{\partial A}{\partial x^4}\,\frac{\partial D}{\partial x^2}
+\frac{1}{CB}\,\frac{\partial C}{\partial x^2}\,\frac{\partial B}{\partial x^4}+\frac{1}{DC}\,\frac{\partial D}{\partial x^2}\,\frac{\partial C}{\partial x^4}\right]
$$

$$
-8\pi C T^3_4 = +8\pi D T^4_3 =
-\frac{1}{2}\left[\frac{1}{A}\,\frac{\partial^2 A}{\partial x^3 \partial x^4}+\frac{1}{B}\,\frac{\partial^2 B}{\partial x^3 \partial x^4}\right]
+\frac{1}{4}\left[\frac{1}{A^2}\,\frac{\partial A}{\partial x^3}\,\frac{\partial A}{\partial x^4}+\frac{1}{B^2}\,\frac{\partial B}{\partial x^3}\,\frac{\partial B}{\partial x^4}+\frac{1}{AC}\,\frac{\partial A}{\partial x^3}\,\frac{\partial C}{\partial x^4}+\frac{1}{AD}\,\frac{\partial A}{\partial x^4}\,\frac{\partial D}{\partial x^3}
+\frac{1}{BC}\,\frac{\partial B}{\partial x^3}\,\frac{\partial C}{\partial x^4}+\frac{1}{BD}\,\frac{\partial B}{\partial x^4}\,\frac{\partial D}{\partial x^3}\right]
$$