

The Flatter Regions of Newman, Unti, and Tamburino's Generalized Schwarzschild Space*

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(Received 17 December 1962)

The "generalized Schwarzschild" metric discovered by Newman, Unti, and Tamburino, which is stationary and spherically symmetric, is investigated. We find that the orbit of a point under the group of time translations is a circle, rather than a line as in the Schwarzschild case. The time-like hypersurfaces $r = \text{const}$ which are left invariant by the group of motions are topologically three-spheres S^3 , in contrast to the topology $S^2 \times R$ (or $S^2 \times S^1$) for the $r = \text{const}$ surfaces in the Schwarzschild case. In the Schwarzschild case, the intersection of a spacelike surface $t = \text{const}$ and an $r = \text{const}$ surface is a sphere S^2 . If σ is any spacelike hypersurface in the generalized metric, then its (two-dimensional) intersection with an $r = \text{const}$ surface is not any closed two-dimensional manifold, that is, the generalized metric admits no reasonable spacelike surfaces. Thus, even though all curvature invariants vanish as $r \rightarrow \infty$, in fact $R_{\mu\nu\alpha\beta} = O(1/r^2)$ as in the Schwarzschild case, this metric is not asymptotically flat in the sense that coordinates can be introduced for which $g_{\mu\nu} - \eta_{\mu\nu} = O(1/r)$. An apparent singularity in the metric at small values of r , which appears to be similar to the spurious Schwarzschild singularity at $r = 2m$, has not been studied. If this singularity should again be spurious, then the "generalized Schwarzschild" space would represent a terminal phase in the evolution of an entirely nonsingular cosmological model which, in other phases, contains closed spacelike hypersurfaces but no matter.

I. INTRODUCTION

THE primary purpose of this paper is to study and describe geometrically the stationary, spherically symmetric solution of the Einstein equations recently discovered by Newman, Unti, and Tamburino¹ which I shall refer to as NUT space. A second important purpose of this work is to provide an example of the recognition and elimination of a spurious singularity in a Riemannian line element with the Lorentz $-+++$ signature. No general method is known for eliminating coordinate singularities in a metric, nor are there adequate criteria to determine that a singularity is not merely a coordinate singularity, and I expect that further examples beyond the Kruskal-Fronsdal^{2,3} elimination of the Schwarzschild singularity will be helpful in leading to an understanding of these problems. A third, minor, aim of this paper is to provide an example of the use of orthonormal frames (tetrads) in a style more economical with indices than is usual in the literature of physics, and in particular a method of computing the curvature tensor very rapidly (cf. Appendix A).

The question of singularities in metrics is broader

and more important than the study of one particular metric to which most of this paper is devoted, so I will briefly summarize the present state of the art.⁴ The first step is to find some clearly stated problems, and the clue to clarity is to refuse ever to speak of a singularity but instead to phrase everything in terms of the properties of differentiable metric fields on manifolds. If one is given a manifold, and on it a metric which does not at all points satisfy the necessary differentiability requirements, one simply throws away all the points of singularity. The starting point for any further discussion is then the largest submanifold on which the metric is differentiable. This is done because there is not known any useful way of describing the singularities of a function except by describing its behavior at regular points near the singularity. The first problem then is to select a criteria which will identify in an intuitively acceptable way a "nonsingular space." Evidently, differentiability is only a minimum prerequisite, since everything becomes differentiable when the singular points are discarded. The problem is rather to recognize the holes left in the space where singular (or even regular) points have been omitted. For a connected Riemannian manifold

* Supported in part by the U. S. Air Force Office of Scientific Research, Air Research and Development Command.

¹ E. Newman, L. Tamburino, and T. Unti, *J. Math. Phys.* **4**, 915 (1963). I wish to thank these authors for sending me a preprint of their paper.

² M. D. Kruskal, *Phys. Rev.* **119**, 1743 (1960).

³ C. Fronsdal, *Phys. Rev.* **116**, 778 (1959).

⁴ I wish to thank Mr. L. Shepley for preparing this review and for correcting numerous errors in an earlier draft. We have borrowed heavily from L. Marcus' lectures on this topic at the American Mathematical Society's 1962 Summer Institute at the University of California at Santa Barbara.

free space-time. As yet, no example is known of a nonflat singularity-free cosmological solution with vanishing cosmological constant. The behavior we are led to conjecture for an extended NUT space would also be quite remarkable. The unequal expansion rates in different directions of the closed spacelike hypersurface $r = 0$ would smoothly develop into a situation where closed spacelike hypersurfaces no longer could exist, while the evolution in time would smoothly resolve itself into a state of affairs which was periodic in time. A difficulty which will arise in attempting to eliminate the Schwarzschild-like singularities at $f^2 = 0$ using Kruskal's methods is the time periodicity of NUT space. For instance, in the Schwarzschild solution for $r > 2m$ (or for $0 < r < 2m$), we can easily identify points to give a periodic time, $t \equiv t + T$. In Kruskal's extended Schwarzschild solution,² we may then attempt to make this same identification. The singularity in the coordinate t causes no difficulty, since it is not the t coordinate, but motions along the Killing vector field $\partial/\partial t$ which define the identifications we wish to make, and $\partial/\partial t$ is an analytic vector field, everywhere free from singularities, as is evident from its expression in terms of Kruskal's nonsingular coordinates u, v :

$$\frac{\partial}{\partial t} = \frac{1}{4m} \left(v \frac{\partial}{\partial u} + u \frac{\partial}{\partial v} \right). \quad (57)$$

What happens, then, when we identify points which differ by a motion of amount T , i.e. when we identify points P and $\exp\{T \partial/\partial t\}P$? In fact, this introduces a singularity, not everywhere along the null 3-surfaces $r = 2m$ (or $u = \pm v$), but only at the single 2-surface $u = v = 0$ where, by Eq. (57), $\partial/\partial t = 0$ so the Killing motion has fixed points.

APPENDIX A. COMPUTATION OF THE CURVATURE

We shall compute the Riemann tensor by methods due to Cartan²⁰ which, at least for metrics with a considerable amount of symmetry, are much more efficient than the methods usually employed by physicists. If $\omega^1, \omega^2, \dots, \omega^n$ are a set of covariant basis vectors, then the metric tensor is written

$$ds^2 = g_{\mu\nu} \omega^\mu \omega^\nu. \quad (A1)$$

Covariant derivatives are formed with the aid of the connection forms $\omega^\mu{}_\nu$, or their components $\Gamma^\mu{}_{\alpha\beta}$:

$$\omega^\mu{}_\alpha = \Gamma^\mu{}_{\alpha\beta} \omega^\beta. \quad (A2)$$

²⁰ E. Cartan [cf. reference 10, Chap. VII]; T. J. Willmore [cf. reference 5, Secs. VII-16 and VII-19]; H. Flanders, Trans. Am. Math. Soc. 75, 311 (1953).

The following two sets of equations determine the $\omega^\mu{}_\alpha$ uniquely:

$$dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu} \quad (A3)$$

$$d\omega^\mu = -\omega^\mu{}_\nu \wedge \omega^\nu. \quad (A4)$$

In the more familiar case of a *coordinate* frame $\omega^\mu = dx^\mu$, the second equation here gives $\Gamma^\mu{}_{\alpha\beta} = \Gamma^\mu{}_{\beta\alpha}$ (using the property $d^2 = 0$ of the exterior derivative, and the antisymmetry of the exterior product \wedge), while the first is a standard relation between the metric derivatives and the $\Gamma_{\mu\alpha\beta}$, which is solved to show that $\Gamma^\mu{}_{\alpha\beta}$ is a Christoffel symbol. We will use these equations in a different case, that of an orthonormal frame, where, since $g_{\mu\nu} = \eta_{\mu\nu} = \text{const}$, Eq. (A3) states that the forms $\omega_{\mu\nu}$ are anti-symmetric

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0. \quad (A5)$$

With the aid of this antisymmetry, Eqs. (A4) can now be solved for $\omega^\mu{}_\nu$, when ω^μ is given. Although a formula like the Christoffel relation exists also in this case, *the computation is most efficient when Eqs. (A) can be solved by inspection*, as we shall shortly illustrate. Once the connection forms have been computed, the curvature forms θ^μ , are obtained from the formula

$$\theta^\mu{}_\nu = d\omega^\mu{}_\nu + \omega^\mu{}_\alpha \wedge \omega^\alpha{}_\nu. \quad (A6)$$

The components of the Riemann tensor $R^\mu{}_{\nu\alpha\beta}$ are then read out of these curvature forms:

$$\theta^\mu{}_\nu = \frac{1}{2} R^\mu{}_{\nu\alpha\beta} \omega^\alpha \wedge \omega^\beta, \quad (A7)$$

and the Ricci tensor is formed by contraction

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \quad (A8)$$

Note that in 4-space, with an *orthonormal* basis ω^μ , there are only six connection forms $\omega^\mu{}_\nu$, in contrast to forty Christoffel symbols, and only six curvature forms θ^μ , in contrast to twenty components $R_{\mu\nu\alpha\beta}$ of the Riemann tensor or ten for the Ricci tensor. In simple cases, such as NUT space, a savings of labor on a scale suggested by these numbers is actually attained, and the Ricci tensor can be computed much more rapidly by these methods (which provide the Riemann tensor as a bonus along the way) than from the usual formula in terms of Christoffel symbols.

The computation begins by writing the metric in terms of an orthonormal frame, as has been done in Eqs. (7) and (8). Next the curl, $d\omega^\mu$, of each base vector ω^μ must be computed. Let us compute, for example, $d\omega^0$ where

$$\omega^0 = f(r)[dt + 4l \sin^2 \frac{1}{2} \theta d\phi]. \quad (A9)$$

First note that $d(f[]) = df \wedge [] + f d[]$, and $df = f' dr = f' f \omega^1$ since $\omega^1 = f^{-1} dr$. Thus $d\omega^0 = f' \omega^1 \wedge \omega^0 + f d[]$. Using $d^2 = 0$ we then find

$$d[] = 4l \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta d\theta \wedge d\phi = 2l \sin \theta d\theta \wedge d\phi = 2l(r^2 + l^2)^{-1} \omega^2 \wedge \omega^3.$$

Proceeding in this manner we obtain

$$\begin{aligned} d\omega^0 &= f' \omega^1 \wedge \omega^0 + 2lf(r^2 + l^2)^{-1} \omega^2 \wedge \omega^3, \\ d\omega^1 &= 0, \\ d\omega^2 &= rf(r^2 + l^2)^{-1} \omega^1 \wedge \omega^2, \\ d\omega^3 &= rf(r^2 + l^2)^{-1} \omega^1 \wedge \omega^3 \\ &\quad + (r^2 + l^2)^{-\frac{1}{2}} \cot \theta \omega^2 \wedge \omega^3. \end{aligned} \tag{A10}$$

Now we must compare this set of equations with Eqs. (A4) and pick out the ω^a . The first equation of (A10), for instance, must take the form

$$d\omega^0 = -\omega^0_1 \wedge \omega^1 - \omega^0_2 \wedge \omega^2 - \omega^0_3 \wedge \omega^3,$$

so we guess that $\omega^0_1 = +f' \omega^0$. By antisymmetry we have $\omega^0_1 = -\omega_{01} = \omega_{10} = \omega^1_0$, and can verify that this choice is consistent with $0 = d\omega^1 = -\omega^1_0 \wedge \omega^0 + \dots$ since $\omega^0 \wedge \omega^0 = 0$. The remaining term in $d\omega^0$ of Eqs. (A10) could arise either from ω^0_2 or ω^0_3 , and this choice can most conveniently be settled later, so we proceed to the $d\omega^2$ equation. From $d\omega^2 = -\omega^2_0 \wedge \omega^0 - \omega^2_1 \wedge \omega^1 - \omega^2_3 \wedge \omega^3$ we guess that $\omega^2_1 = rf(r^2 + l^2)^{-1} \omega^2$, so that ω^2_0 and ω^2_3 terms must cancel here. In this manner, one proceeds to fill out the list below. As the solution ω^a , of Eqs. (A4) and (A5) is known to be unique, the proof that a guess for a set of ω^a , is correct is simply that it satisfies these equations. We have then

$$\begin{aligned} \omega^0_1 &= +\omega^1_0 = f' \omega^0, \\ \omega^0_2 &= +\omega^2_0 = lf(r^2 + l^2)^{-1} \omega^3, \\ \omega^0_3 &= +\omega^3_0 = -lf(r^2 + l^2)^{-1} \omega^2, \\ \omega^2_3 &= -\omega^3_2 = lf(r^2 + l^2)^{-1} \omega^0 - (r^2 + l^2)^{-\frac{1}{2}} \cot \theta \omega^3, \\ \omega^3_1 &= -\omega^1_3 = rf(r^2 + l^2)^{-1} \omega^3, \\ \omega^1_2 &= -\omega^2_1 = -rf(r^2 + l^2)^{-1} \omega^2. \end{aligned} \tag{A11}$$

Of a possible 24 connection components $\Gamma^a_{\alpha\beta}$, only the seven nonvanishing ones which appear via Eq. (A2) in Eqs. (A11) caused us any labor. That the others vanish we discovered by finding no need for additional terms in Eqs. (A11), not by explicitly evaluating a formula for $\Gamma^a_{\alpha\beta}$ and obtaining a zero result. The curvature computation is now purely mechanical. The first of Eqs. (A6), for instance, reads

$$\theta^0_1 = d\omega^0_1 + \omega^0_2 \wedge \omega^2_1 + \omega^0_3 \wedge \omega^3_1. \tag{A12}$$

Only two terms appear in the sum because of the antisymmetry of $\omega_{\mu\nu}$. Substituting in this and similar formulas from Eq. (A11) yields

$$\begin{aligned} \theta_{01} &= -2A\omega^0 \wedge \omega^1 - 2D\omega^2 \wedge \omega^3, \\ \theta_{02} &= +C\omega^0 \wedge \omega^2 + D\omega^3 \wedge \omega^1, \\ \theta_{03} &= +C\omega^0 \wedge \omega^3 + D\omega^1 \wedge \omega^2, \\ \theta_{23} &= +2B\omega^2 \wedge \omega^3 - 2D\omega^0 \wedge \omega^1, \\ \theta_{31} &= -C\omega^3 \wedge \omega^1 + D\omega^0 \wedge \omega^2, \\ \theta_{12} &= -C\omega^1 \wedge \omega^2 + D\omega^0 \wedge \omega^3, \end{aligned} \tag{A13}$$

where

$$\begin{aligned} A &= \frac{1}{4} (f^2)''', \\ B &= \frac{1}{2} [-f^2 + 1 + 4l^2 f^2 / (r^2 + l^2)] (r^2 + l^2)^{-1}, \\ C &= [\frac{1}{2} r (f^2)' + l^2 f^2 / (r^2 + l^2)] (r^2 + l^2)^{-1}, \\ D &= [\frac{1}{2} l (f^2)' - r l f^2 / (r^2 + l^2)] (r^2 + l^2)^{-1}. \end{aligned} \tag{A14}$$

The first line of Eqs. (A13), for example, tells us, by comparison with Eq. (A6), that $R_{0101} = -2A$, $R_{0123} = -2D$, $R_{0102} = 0$, $R_{0131} = 0$, etc. The contractions necessary to form $R_{\mu\nu}$, for instance

$$\begin{aligned} R_{11} &= R_{1^0 10} + R_{1^2 12} + R_{1^3 13} \\ &= -R_{1010} + R_{1212} + R_{1313} \\ &= 2A - 2C, \end{aligned}$$

are readily performed by scanning Eqs. (A13). Thus we find

$$R \equiv R^a_a \equiv R^{\mu\nu}_{\mu\nu} = 4(A + B - 2C), \tag{A15}$$

while for

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

the only nonvanishing components are

$$\begin{aligned} G_{11} &= -G_{00} = 2(C - B), \\ G_{22} &= G_{33} = 2(C - A). \end{aligned} \tag{A16}$$

The empty-space Einstein equations thus require $A = B = C$.

Each of the quantities A, B, C , and D is effectively an invariant since the basis vectors ω^0 and ω^1 we used can be characterized geometrically, while the form (A13) of the curvature tensor is invariant under rotations in the 23 plane. We can characterize the vector ω^1 as the unit normal to the orbits $r = \text{const}$ of the group of motions; similarly, the contravariant vector e_0 from the dual basis can be characterized as the unit vector parallel to that unique Killing vector which commutes with all the Killing vectors. These four invariants reduce

to two when the Einstein equations are satisfied, and for the Schwarzschild solution, where $l = 0$, evidently $D = 0$ so there remains only one invariant, $A = m/r^3$. For NUT space we have

$$A = B = C = (r^2 + l^2)^{-3} \times (mr^3 + 3l^2r^2 - 3ml^2r - l^4), \tag{A17}$$

$$D = -l(r^2 + l^2)^{-3}(r^3 - 3mr^2 - 3l^2r + ml^2).$$

APPENDIX B. VECTOR FIELDS ON S^3

We collect here some well known properties of S^3 , including a computation of the commutation relations in Eq. (35). As coordinates for a quaternion q , we take $w(q), x(q), y(q), z(q)$, defined by

$$q = w + ix + jy + kz. \tag{B1}$$

The unit quaternions satisfying

$$|q|^2 \equiv qq^* \equiv w^2 + x^2 + y^2 + z^2 = 1 \tag{B2}$$

constitute S^3 . They form a group under multiplication since $|qp| = |q| \cdot |p|$. This group S^3 can be related to the rotation group $SO(3)$ as follows: Identify the imaginary quaternions r which satisfy $r + r^* = 0$ [i.e. $w(r) = 0$] with Euclidean 3-space R^3 . Then the transformations

$$r \rightarrow qrq^* \tag{B3}$$

map R^3 into itself [check $qrq^* + (qrq^*)^* = 0$ using the rule $(pq)^* = q^*p^*$] and, when $|q| = 1$, preserve $|r|$ which is just the Euclidean norm since $w(r) = 0$. Both q and $-q$ give the same transformation (B3), so S^3 covers $SO(3)$ twice.

Both the transformations

$$q \rightarrow pq \tag{B4\xi}$$

and

$$q \rightarrow qp^*, \tag{B4\eta}$$

where $|p| = 1$ preserve the quaternion norm $|q|$ and therefore map S^3 into itself. The infinitesimal generators of these transformations are respectively the vector fields ξ_a and \mathbf{n}_a defined in Table I. For instance ξ_x generates the one-parameter subgroup

of (B4\xi) transformations obtained by taking $p = e^{i\alpha/2}$. We now want to represent ξ_x in terms of its components along the base vectors $\partial_w = \partial/\partial w$, etc. associated with the coordinate system we are using. Differentiation of $f(q)$ along ξ_x is defined by

$$\xi_x[f] = [(d/d\alpha)f(e^{i\alpha/2}q)]_{\alpha=0}. \tag{B5}$$

TABLE I. The vector fields ξ_a and \mathbf{n}_a on S^3 are defined as the infinitesimal generators $\partial/\partial\alpha$ of certain one-parameter groups of motions of S^3 as shown.

Group of motions	One-parameter subgroup		
	$p = e^{i\alpha/2}$	$p = e^{j\alpha/2}$	$p = e^{k\alpha/2}$
left translations $q \rightarrow pq$	ξ_x	ξ_y	ξ_z
right translations $q \rightarrow qp^*$	\mathbf{n}_x	\mathbf{n}_y	\mathbf{n}_z

To compute the components of ξ_x , namely $\xi_x[w]$ etc., we note that $d(e^{i\alpha/2}q)/d\alpha = (\frac{1}{2})iq$; so, upon taking the various real and imaginary parts of this relation we find $\xi_x[w] \equiv \xi_x[\text{Re } q] = \text{Re}[(\frac{1}{2})iq] = -\frac{1}{2}x$, etc. The results of this and similar computations are summarized as follows:

$$\begin{aligned} 2\xi_x &= -x\partial_w + w\partial_x - z\partial_y + y\partial_z, \\ 2\xi_y &= -y\partial_w + z\partial_x + w\partial_y - x\partial_z, \\ 2\xi_z &= -z\partial_w - y\partial_x + x\partial_y + w\partial_z, \\ 2\mathbf{n}_x &= +x\partial_w - w\partial_x - z\partial_y + y\partial_z, \\ 2\mathbf{n}_y &= +y\partial_w + z\partial_x - w\partial_y - x\partial_z, \\ 2\mathbf{n}_z &= +z\partial_w - y\partial_x + x\partial_y - w\partial_z. \end{aligned} \tag{B6}$$

The commutators of these differential operators are now straightforwardly computed with the result given in Eq. (35). For these computations, we have regarded ξ_a and \mathbf{n}_a as defined over the entire space R^4 of all quaternions, which permitted the convenience of using rectangular coordinates. Since the results [Eqs. (35)], do not refer to any coordinate system, and since all the vectors involved are tangent to S^3 , the method of extending the definition of the vector fields beyond S^3 was irrelevant.

