

# LECTURE 39

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- I. Curvature of a two-sphere.
- II. Ricci tensor and the curvature invariant
- III. Coordinate vs. orthonormal basis components

I. To illustrate the calculation of metric compatible curvature via the 39.2 Cartan-Misner method of differential forms, consider a two-sphere of radius  $a$ . The method is a three step process. It starts with the metric.

① For the given metric of a sphere,

$$g = ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\varphi^2; \quad [g_{\mu\nu}] = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{bmatrix}$$

introduce an orthonormal basis of covectors:

$$\begin{aligned} g = ds^2 &= (a d\theta)^2 + (a \sin\theta d\varphi)^2 \\ &= (\hat{\omega}^\theta)^2 + (\hat{\omega}^\varphi)^2 \quad ; \end{aligned}$$

This basis is

$$\begin{aligned} \hat{\omega}^\theta &= a d\theta \\ \hat{\omega}^\varphi &= a \sin\theta d\varphi \end{aligned}$$

Note bene.

(i) From the dual basis one infers that the orthonormal basis is

$$\left\{ \hat{e}_\theta = \frac{1}{a} \frac{\partial}{\partial \theta} \equiv \frac{1}{a} e_\theta, \quad \hat{e}_\varphi = \frac{1}{a \sin\theta} \frac{\partial}{\partial \varphi} \equiv \frac{1}{a \sin\theta} e_\varphi \right\}$$

and hence

$$[\hat{g}_{\mu\nu}] = [\hat{e}_\mu \cdot \hat{e}_\nu] = [\delta_{\mu\nu}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(ii) The starting point for the Cartan-Misner method is the dual basis.

②. Calculate the connection one-forms by using

$$d\hat{\omega}^\mu + \hat{\omega}^\nu \wedge \hat{\omega}^\gamma = 0 \quad (1^{\text{st}} \text{ structure equation})$$

$$\hat{\omega}_{\mu\nu} + \hat{\omega}_{\nu\mu} = d\hat{g}_{\mu\nu} \quad (\text{metric compatibility})$$

The metric coefficients  $\hat{g}_{\mu\nu} = \hat{e}_\mu \cdot \hat{e}_\nu$  are constants relative to an orthonormal basis. Thus,

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$$d\hat{g}_{\mu\nu} = 0.$$

The system of linear equations for the connection one-forms are

$$\hat{\omega}^\nu \wedge \hat{\omega}^\mu = d\hat{\omega}^\mu$$

$$\hat{\omega}_{\mu\nu} = -\hat{\omega}_{\nu\mu}$$

$$(a) \quad d\hat{\omega}^\theta = 0$$

$$d\hat{\omega}^\varphi = d(a \sin\theta d\varphi) = a \cos\theta d\theta \wedge d\varphi$$

$$= a \cos\theta d\theta \wedge d\varphi$$

$$= \hat{\omega}^\theta \wedge \frac{a \cos\theta}{a \sin\theta} \hat{\omega}^\varphi$$

(b)

$$(i) \quad d\hat{\omega}^\theta + \hat{\omega}^\theta_\theta \wedge \hat{\omega}^\theta + \hat{\omega}^\theta_\varphi \wedge \hat{\omega}^\varphi = 0$$

$$\boxed{\hat{\omega}^\theta_\theta = \hat{\omega}_{\theta\theta} = 0}$$

$$\therefore \hat{\omega}^\theta_\varphi \wedge \hat{\omega}^\varphi = 0$$

$$\hat{\omega}^\theta_\varphi = (?) \omega^\varphi$$

$$(ii) \quad d\hat{\omega}^\varphi + \hat{\omega}^\varphi_\theta \wedge \hat{\omega}^\theta + \hat{\omega}^\varphi_\varphi \wedge \hat{\omega}^\varphi = 0$$

$$\boxed{\hat{\omega}^\varphi_\varphi = \hat{\omega}_{\varphi\varphi} = 0}$$

$$\hat{\omega}^\theta_\varphi \wedge \hat{\omega}^\varphi_\theta = \frac{\cot\theta}{a} \hat{\omega}^\theta_\varphi \wedge \hat{\omega}^\varphi$$

$$\therefore \hat{\omega}^\varphi_\theta = \frac{\cot\theta}{a} \hat{\omega}^\varphi + (??) \hat{\omega}^\theta$$

Compare this with b)(ii). This is possible because

$$\hat{\omega}^\varphi_\theta = \hat{\omega}_{\varphi\theta} = -\hat{\omega}_{\theta\varphi} = -\hat{\omega}^\theta_\varphi$$

Thus

$$\omega^\varphi_\theta = \frac{\cot\theta}{a} \hat{\omega}^\varphi + (??) \hat{\omega}^\theta = -(?) \omega^\varphi$$

$$(??) = 0$$

$$(?) = -\frac{\cot\theta}{a}$$

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$$\begin{cases} \hat{\omega}^\varphi_\theta = \frac{\cot\theta}{a} \hat{\omega}^\varphi \\ \hat{\omega}^\theta_\varphi = -\frac{\cot\theta}{a} \hat{\omega}^\varphi \end{cases}$$

c) Conclusion

$$[\hat{\omega}^\mu_\nu] = \begin{bmatrix} \hat{\omega}^\theta_\theta = 0 & \hat{\omega}^\theta_\varphi = -\frac{\cot\theta}{a} \hat{\omega}^\varphi \\ \hat{\omega}^\varphi_\theta = \frac{\cot\theta}{a} \hat{\omega}^\varphi & \hat{\omega}^\varphi_\varphi = 0 \end{bmatrix}$$

③ Calculate curvature 2-form

$$\hat{\Omega}^\mu_\nu = d\hat{\omega}^\mu_\nu + \hat{\omega}^\mu_\gamma \wedge \hat{\omega}^\gamma_\nu$$

$$\begin{aligned} (i) \quad \hat{\Omega}^\theta_\varphi &= d\hat{\omega}^\theta_\varphi + \hat{\omega}^\theta_\theta \wedge \hat{\omega}^\theta_\varphi + \hat{\omega}^\theta_\varphi \wedge \hat{\omega}^\varphi_\varphi \\ &= d\left(-\frac{\cot\theta}{a} \hat{\omega}^\varphi\right) + 0 + 0 \end{aligned}$$

Nota bene.

One could use the product rule to calculate this differential:

$$d\left(-\frac{\cot\theta}{a} \hat{\omega}^\varphi\right) = -d\left(\frac{\cot\theta}{a}\right) \wedge \hat{\omega}^\varphi - \frac{\cot\theta}{a} d(\hat{\omega}^\varphi)$$

However, it is easier to revert to the coordinate basis, take the differential, and then reintroduce the orthonormal basis after that.

With that in mind, the calculation yields

$$\begin{aligned} \hat{\Omega}^\theta_\varphi &= -d\left(\frac{1}{a} \frac{\cos\theta}{\sin\theta} a \sin\theta d\varphi\right) \\ &= -d(\cos\theta d\varphi) \end{aligned}$$

$$\begin{aligned}
 &= \sin \theta \, d\theta \wedge d\varphi \\
 &= \frac{1}{a^2} \hat{\omega}^\theta \wedge \hat{\omega}^\varphi
 \end{aligned}$$

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(ii) Calculate the remaining components in the same way:

$$\begin{aligned}
 \hat{\Sigma}^\varphi_\theta &= d\hat{\omega}^\varphi_\theta + \hat{\omega}^\varphi_\theta \wedge \hat{\omega}^\theta_\theta + \hat{\omega}^\varphi_\varphi \wedge \hat{\omega}^\theta_\theta \\
 &= d\left(\frac{\cot \theta}{a} \omega^\varphi\right) \\
 &= \frac{-1}{a^2} \hat{\omega}^\theta \wedge \omega^\varphi
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \hat{\Sigma}^\theta_\theta &= d\hat{\omega}^\theta_\theta + \hat{\omega}^\theta_\theta \wedge \hat{\omega}^\theta_\theta + \hat{\omega}^\theta_\varphi \wedge \hat{\omega}^\varphi_\theta \\
 &= 0 + 0 + 0 \\
 \hat{\Sigma}^\varphi_\varphi &= 0
 \end{aligned}$$

(iv) Use the metric induced antisymmetry in the curvature as a computational check

$$\hat{\Sigma}^\theta_\varphi = \hat{\Sigma}_{\theta\varphi} = -\hat{\Sigma}_{\varphi\theta} = -\hat{\Sigma}^\varphi_\theta,$$

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which agrees with the calculations.

Conclusion

$$\begin{bmatrix} \hat{\Sigma}^\mu_\nu \end{bmatrix} = \begin{bmatrix} \hat{\Sigma}^\theta_\theta = 0 & \hat{\Sigma}^\theta_\varphi = \frac{1}{a^2} \hat{\omega}^\theta \wedge \omega^\varphi \\ \hat{\Sigma}^\varphi_\theta = \frac{-1}{a} \hat{\omega}^\theta \wedge \omega^\varphi & \hat{\Sigma}^\varphi_\varphi = 0 \end{bmatrix}$$

Remark.

The hatted elements are the coefficients of  $\hat{e}_\mu \otimes \hat{\omega}^\nu \hat{\Sigma}^\mu_\nu$ , relative to a physical, i.e. orthonormal, basis for each

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tangent space  $T_p(S^2)$  on the two-sphere. For such a basis the rotational symmetry of  $S^2$  implies that the curvature is independent of  $\theta$  and  $\varphi$  on  $S^2$ .

Read out the non-zero components of the curvature tensor from the curvature 2-form (the 2<sup>nd</sup> structure equation):

$$\begin{aligned}\hat{\Sigma}^\mu{}_\nu &= \hat{R}^\mu{}_{\nu|\alpha\beta} \hat{\omega}^\alpha \hat{\omega}^\beta = \hat{R}^\mu{}_{\nu\theta\varphi} \hat{\omega}^\theta \hat{\omega}^\varphi \\ &= \frac{1}{2} \hat{R}^\mu{}_{\nu\alpha\beta} \hat{\omega}^\alpha \wedge \hat{\omega}^\beta = \frac{1}{2} (\hat{R}^\mu{}_{\nu\theta\varphi} - \hat{R}^\mu{}_{\nu\varphi\theta}) \hat{\omega}^\theta \wedge \hat{\omega}^\varphi \\ \hat{\Sigma}^\theta{}_\varphi &= \frac{1}{2} \hat{R}^\theta{}_{\varphi\theta\varphi} \hat{\omega}^\theta \wedge \hat{\omega}^\varphi + \frac{1}{2} \hat{R}^\theta{}_{\varphi\varphi\theta} \hat{\omega}^\varphi \wedge \hat{\omega}^\theta \\ \frac{1}{a^2} \hat{\omega}^\theta \wedge \hat{\omega}^\varphi &= R^\theta{}_{\varphi\theta\varphi} \hat{\omega}^\theta \wedge \hat{\omega}^\varphi\end{aligned}$$

$$\therefore \boxed{R^\theta{}_{\varphi\theta\varphi} = \frac{1}{a^2}}$$

From the antisymmetry of  $R_{\mu\nu\alpha\beta}$  under the interchange  $\mu \leftrightarrow \nu$  this is the only independent non-zero component of the curvature tensor.

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## II Ricci tensor and curvature invariant

The curvature tensor is one of rank (3). The rank of such a tensor is lowered by the contraction map highlighted on page 16.8 of Lecture 18. The result is the Ricci tensor, which is of rank (2):

$$R^{\beta}_{\alpha\beta\gamma} \equiv R_{\alpha\gamma}.$$

This tensor is symmetric,  $R_{\alpha\gamma} = R_{\gamma\alpha}$ . This is a consequence of

$$R^{\delta}_{\alpha\beta\gamma} + R^{\delta}_{\beta\gamma\alpha} + R^{\delta}_{\gamma\alpha\beta} = T^{\delta}_{\alpha\beta;\gamma} + T^{\delta}_{\beta\gamma;\alpha} + T^{\delta}_{\gamma\alpha;\beta}$$

whenever the torsion-based right hand side vanishes.

For the two-sphere the Ricci tensor components are

$$\hat{R}_{\theta\theta} = \hat{R}^{\theta}_{\theta\theta\theta} + \hat{R}^{\varphi}_{\theta\varphi\theta} = \frac{1}{a^2}$$

$$\hat{R}_{\theta\varphi} = \hat{R}^{\theta}_{\theta\theta\varphi} + \hat{R}^{\varphi}_{\theta\varphi\varphi} = 0$$

$$\hat{R}_{\varphi\theta} = \hat{R}^{\theta}_{\varphi\theta\theta} + \hat{R}^{\varphi}_{\varphi\varphi\theta} = 0$$

$$\hat{R}_{\varphi\varphi} = \hat{R}^{\theta}_{\varphi\theta\varphi} + \hat{R}^{\varphi}_{\varphi\varphi\varphi} = \frac{1}{a^2}$$

$$\boxed{[\hat{R}_{\mu\nu}] = \begin{bmatrix} \hat{R}_{\theta\theta} & \hat{R}_{\theta\varphi} \\ \hat{R}_{\varphi\theta} & \hat{R}_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}}$$

The curvature invariant for the two-sphere is

$$R = \hat{R}^{\theta}_{\theta} + \hat{R}^{\varphi}_{\varphi} = \frac{2}{a^2}$$

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### III. Coordinate vs orthonormal basis components

The coordinate basis components of the curvature tensor are determined from its o.n. basis components by following the reasoning process as illustrated with the

$$\text{1-form } \underline{A} = \hat{A}_\mu \hat{\omega}^\mu = \sum_\mu \underbrace{\hat{A}_\mu |g_{\mu\mu}|^{1/2}}_{A_\mu} dx^\mu = \sum A_\mu dx^\mu.$$

Thus

$$\hat{A}_\theta \hat{\omega}^\theta + \hat{A}_\varphi \hat{\omega}^\varphi = \hat{A}_\theta a d\theta + \hat{A}_\varphi a \sin\theta d\varphi$$

implies

$$\begin{aligned} A_\theta &= a \hat{A}_\theta \\ A_\varphi &= a \sin\theta \hat{A}_\varphi. \end{aligned}$$

Apply this line of reasoning to each of curvature and Ricci tensor components. The result is

$$\begin{aligned} R_{\theta\varphi\theta\varphi} &= (a)(a \sin\theta)(a)(a \sin\theta) \hat{R}_{\theta\varphi\theta\varphi} \\ &= a \sin^2\theta \end{aligned}$$

$$R_{\theta\theta} = (a)(a) \hat{R}_{\theta\theta} = 1$$

$$R_{\varphi\varphi} = (a \sin\theta)(a \sin\theta) \hat{R}_{\varphi\varphi} = \sin^2\theta$$



$$\left[ \hat{R}_{\mu\nu} \right] = \begin{bmatrix} R_{\theta\theta} & R_{\theta\varphi} \\ R_{\varphi\theta} & R_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

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which is obviously different from

$$\left[ \hat{R}_{\mu\nu} \right] = \begin{bmatrix} \hat{R}_{\theta\theta} & \hat{R}_{\theta\varphi} \\ \hat{R}_{\varphi\theta} & \hat{R}_{\varphi\varphi} \end{bmatrix} = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2} \end{bmatrix}$$